Mathematical Problems in Climate Dynamics, CIMA + IFAECI

Univ. of Buenos Aires, 2–13 November 2018

Basic Facts of GFD + Atmospheric LFV, Wind-driven Oceans, Paleoclimate & "Tipping Points"

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Please visit these sites for more info.

https://dept.atmos.ucla.edu/tcd, http://www.environnement.ens.fr/
and https://www.researchgate.net/profile/Michael_Ghil

Overall Outline

- **Lecture I: Observations and planetary flow theory (GFD[⌘])**
- **Lecture II: Atmospheric LFV[*] & LRF[**]**
- **Lecture III: EBMs[+], paleoclimate & "tipping points"**
- **Lecture IV: Nonlinear & stochastic models—RDS[❖]**
- **Lecture V: Advanced spectral methods–SSA[±] *et al.***
- **Lecture VI: The wind-driven ocean circulation**

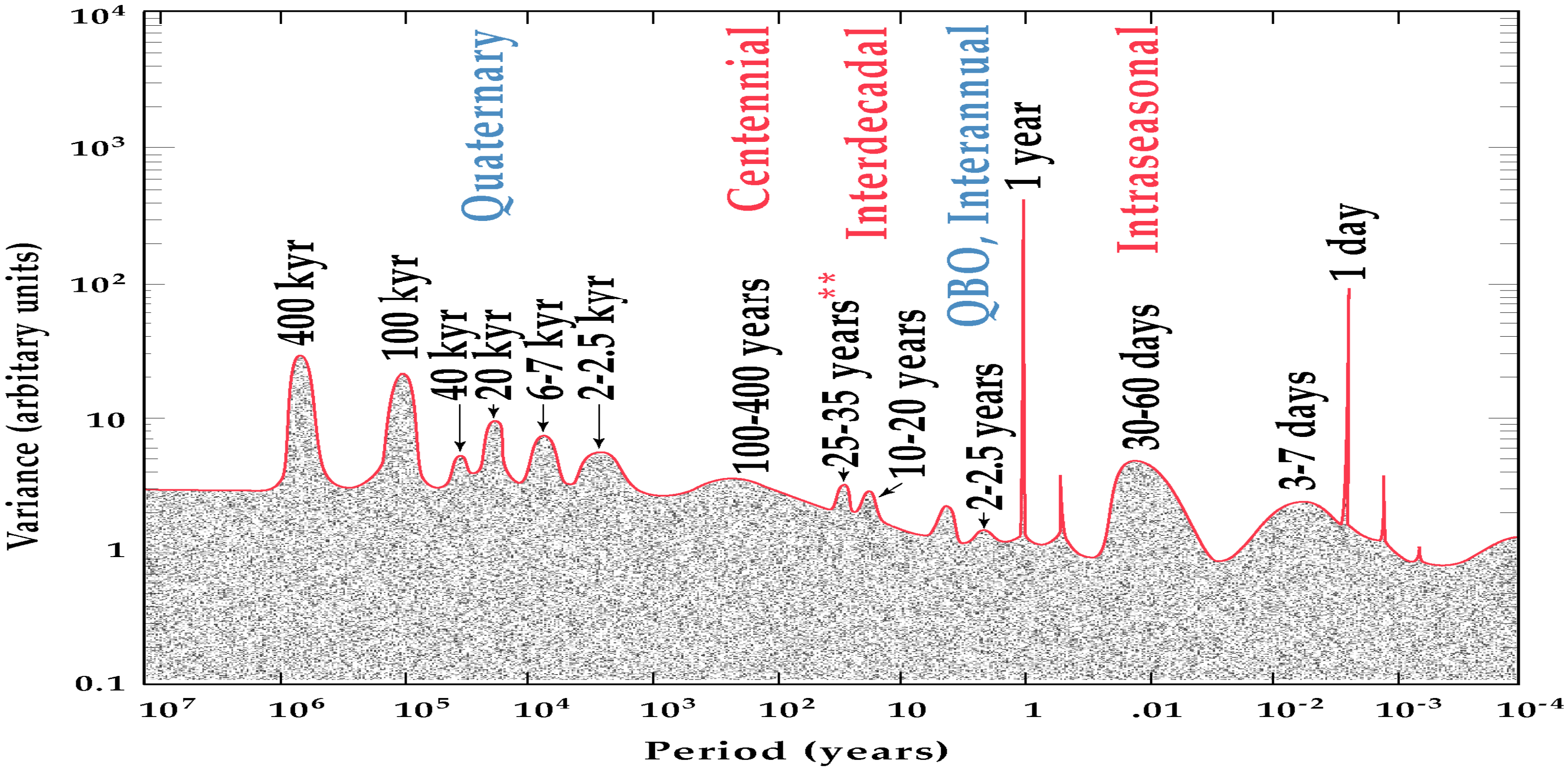
[⌘] GFD = Geophysical fluid dynamics
[*] LFV = Low-frequency variability
[**] LRF = Long-range forecasting
[+] EBM = Energy balance model
[❖] RDS = Random dynamical system
[±] SSA = Singular-spectrum analysis

Composite spectrum of climate variability

Standard treatement of frequency bands:

1. High frequencies – white noise (or "colored")

2. Low frequencies – slow evolution of parameters

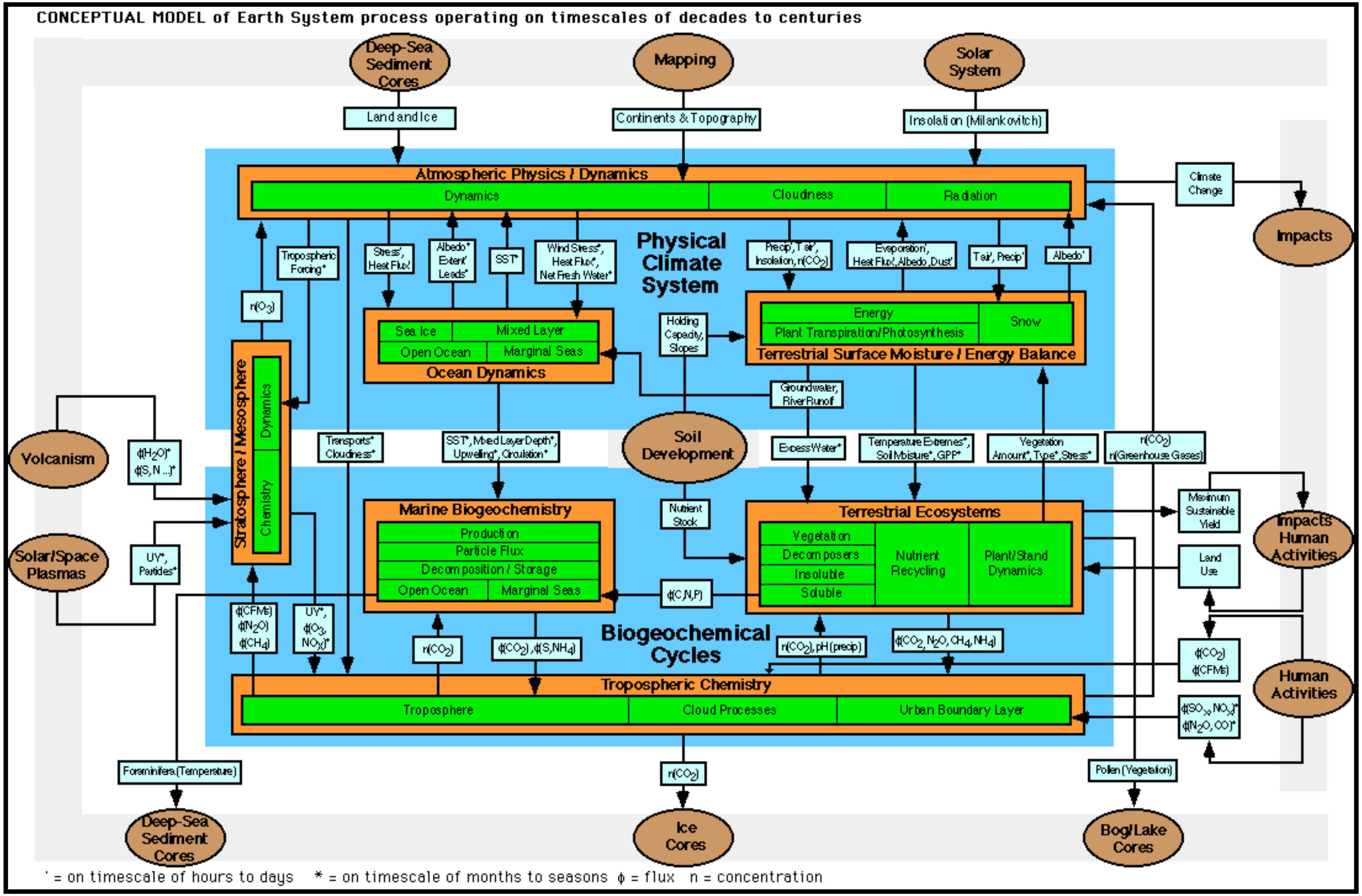


From **Ghil (2001, *EGEC*)**, after **Mitchell* (1976)**

*** "No known source of deterministic internal variability"**

**** 27 years – Brier (1968, *Rev. Geophys.*)**

F. Bretherton's "horrendogram" of Earth System Science

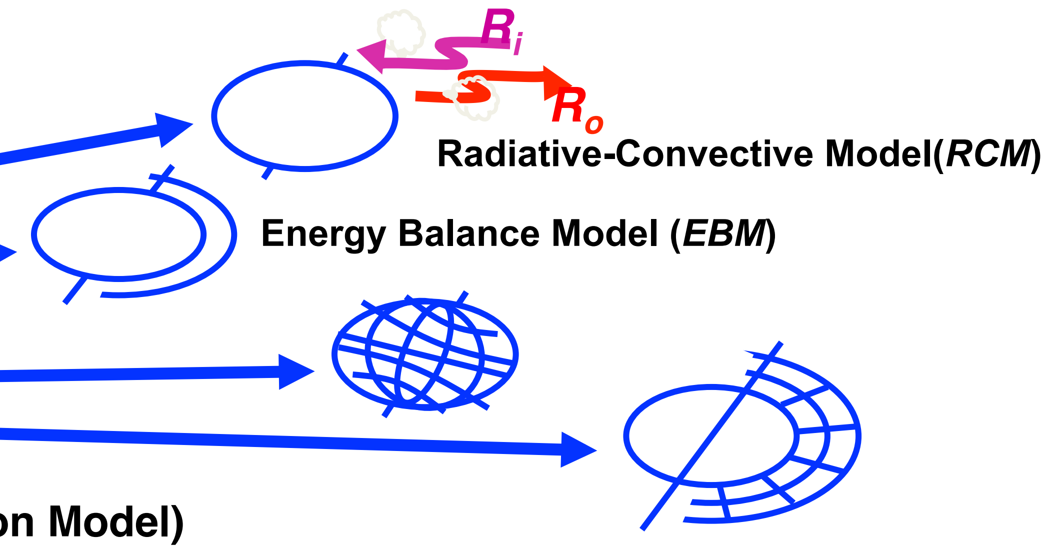


Earth System Science Overview, *NASA Advisory Council*, 1986

Climate models (atmospheric & coupled) : A classification

- *Temporal*
  - stationary, (quasi-)equilibrium
  - transient, climate variability

- *Space*
  - 0-D (dimension 0)
  - 1-D
    - vertical
    - latitudinal
  - 2-D
    - horizontal
    - meridional plane
  - 3-D, GCMs (General Circulation Model)
  - Simple and intermediate 2-D & 3-D models



- *Coupling*
  - Partial
    - unidirectional
    - asynchronous, hybrid
  - Full

➔ ***Hierarchy***: back-and-forth between the simplest and the most elaborate model, and between the models and the observational data

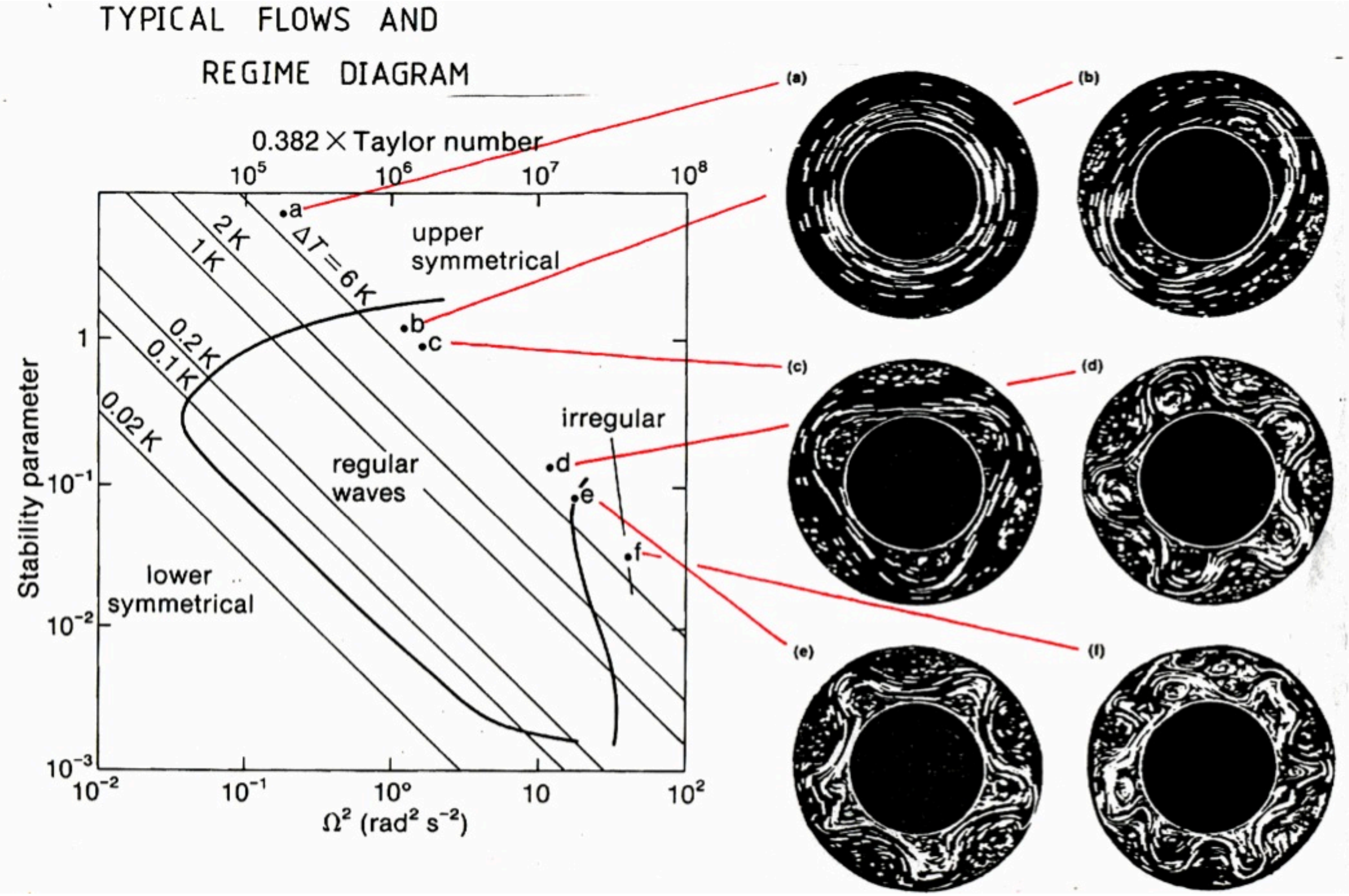
Motivation

- There's a lot of talk about "tipping points."
- It sounds threatening, like falling off a cliff: that's why we care!
- But what are they, and what do **we know** about them?
- Here's a **disambiguation page** (cf. Wikipedia), first.
- *Sociology*: "the moment of critical mass, the threshold, the boiling point" (Gladwell, 2000); a previously rare phenomenon becomes rapidly and dramatically more common.
- *Physics*: the point at which a system changes from a stable equilibrium into a new, qualitatively dissimilar equilibrium (throwing a switch, tilting a plank, boiling water, etc.).
- *Climatology*: "A climate tipping point is a somewhat ill-defined concept [...]"— so we'll try to actually define it better.

- *Catastrophe theory*: branch of bifurcation theory in the study of dynamical systems; here, a tipping point is "a parameter value at which the set of equilibria abruptly change." ➔ **Let's see!**

M. Gladwell (2000) *The Tipping Point: How Little Things Can Make a Big Difference.*
T. M. Lenton *et al.* (2008) Tipping elements in the Earth's climate system, *PNAS,* v. **105.**

Rotating Convection: An Illustration



M. Ghil, P.L. Read & L.A. Smith (*Astron. Geophys., 2010)*

Outline, Tipping Points I

Elementary Bifurcation Theory and Variational Principle

1. Fixed Points
   - linear stability
   - non-linear stability and attractor basins
2. Saddle-node bifurcations
   - multiple branches of stationary solutions
   - linear stability
3. Bifurcations in 1-D
4. Non-linear stability and variational principle
   - variational principle in 0-D
   - variational principle in 1-D
5. Bistability and hysteresis

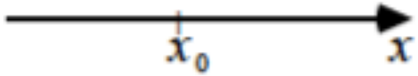
1. Fixed points, I

We start with a scalar ordinary differential equation (ODE)

$$\dot{x} = f(x; \mu)$$

depending on the parameter μ.

Linear stability, $\mu = 1$.



$$f(x_0) = 0 \Rightarrow \dot{x} = 0 \Rightarrow x \equiv x_0$$ – *Fixed point (FP)*

Consider an initial perturbation at $t = 0$:

$$x(0) = x_0 + \xi(0),$$

$$\dot{x} = \dot{x}_0 + \dot{\xi} = \dot{\xi}$$

$$= f(x_0 + \xi) = f(x_0) + f'(x_0)\xi + O(\xi^2)$$

For an infinitesimal perturbation $\xi(0) = \xi_0$

$$\dot{\xi} = f'(x_0)\xi, \qquad f'(x_0) = \lambda, \qquad \dot{\xi} = \lambda\xi,$$

$$\Rightarrow \xi(t) = e^{\lambda t}\xi(0)$$

1. Fixed points, II

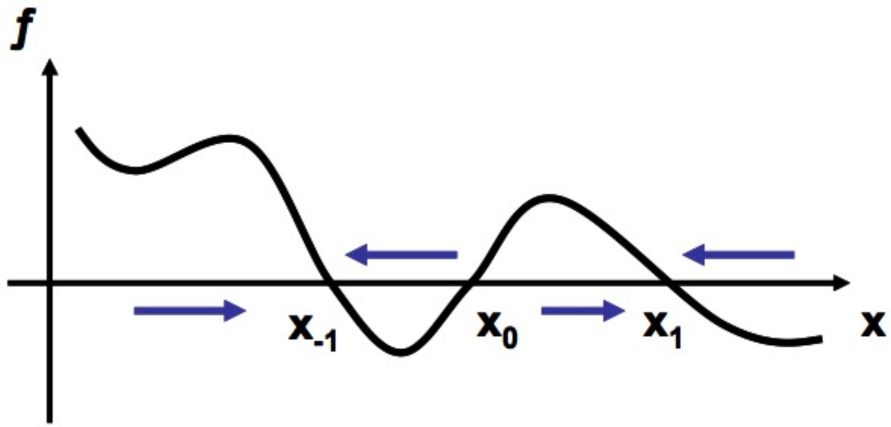
If $\lambda < 0 \Rightarrow$ the fixed point (FP) is (linearly) stable

If $\lambda > 0 \Rightarrow$ the FP is (linearly) unstable

If $\lambda = 0 \Rightarrow$ the linear stability of the FP is neutral

Some basic features on FPs:

1. $f \in C^1, f \not\equiv 0$ on all sub-intervals: FPs are isolated (generic property)
2. Basins of attraction are open intervals (possibly semi-infinite)



2. Saddle-node bifurcations

How does the geometry of the solutions change when $\mu \neq \mu_0$, i.e. how do the number of the stability of the stationary solution change?

Let us start with the scalar case.

A simple case: the saddle-node

$$\dot{x} = \mu - x^2 \equiv f(x;\mu)$$

FPs: $\mu - x^2 = 0$. $x = \pm\sqrt{\mu}$

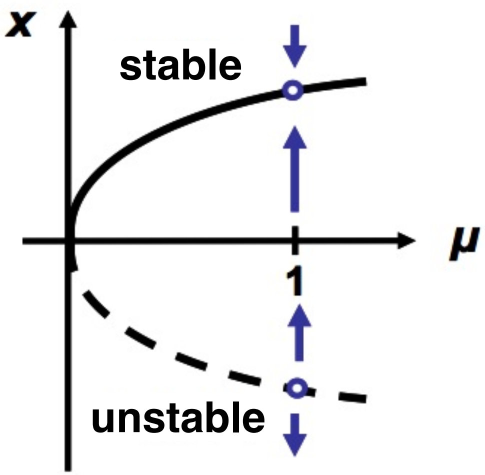
FP stability:

$$x_1 = \sqrt{\mu}, \quad x_{-1} = -\sqrt{\mu}$$

$$x(0) = x_{\pm 1} + \xi(0)$$

$$\dot{\xi} = \lambda_{\pm}\xi,$$

$$\lambda_{\pm} \equiv f'(x_{\pm 1}) = -2x_{\pm 1} = \mp 2\sqrt{\mu}$$



Upper branch (linearly) *stable*

Lower branch (linearly) *unstable*

Let us now examine the nonlinear stability

3. Bifurcations in *n*-D

We studied the scalar case (*n* = 1). More generally, we have:

$\dot{\mathbf{x}} = f(\mathbf{x};\mu)$, $f \in C(\Re^n \times \Re)$, with $\mathbf{x} \in \Re^n$ and $\mu \in \Re$.

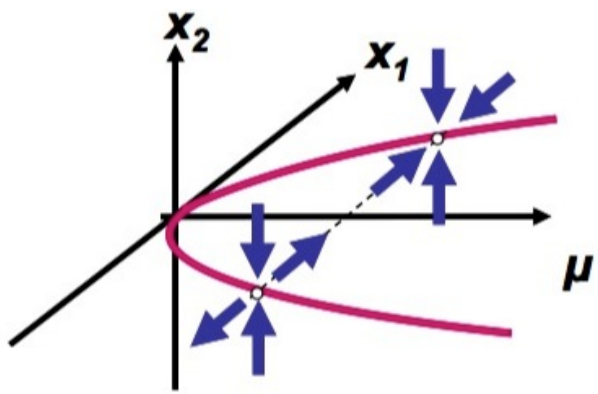
The behavior is "almost" linear in all the phase-parameter space $\Re^n \times \Re$, except in the neighborhood of a few isolated points (x_c, μ_c) : these are bifurcation points, where the Jacobian matrix $L = (\partial f_i / \partial x_j)$ is singular, i.e. $\det L = 0$

In the case n = 2, we can reduce to the normal form :

$$\dot{x_1} = \mu - x_1^2$$
$$\dot{x_2} = -\lambda x_2, \ \lambda > 0$$

In the general case, the reduction gives :

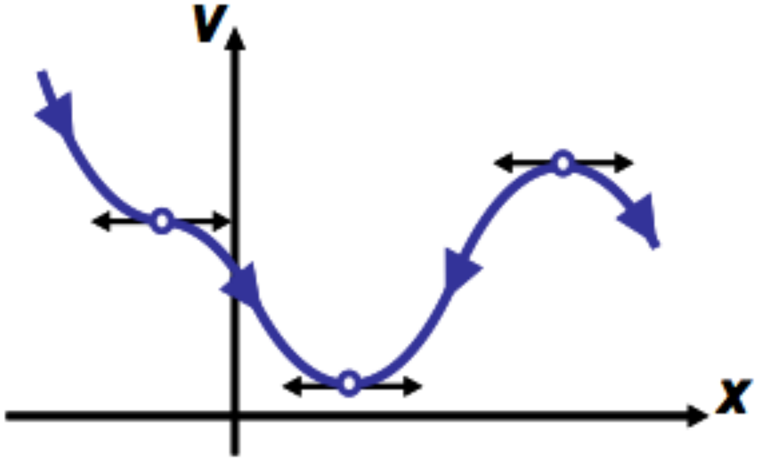
$$\dot{x_1} = \mu - x_1^2$$
$$\dot{x_i} = -\lambda_i x_i, \ \lambda_i > 0, \ i = 2, ..., n$$



This shape explains the “saddle-node bifurcation” terminology

4. Non-linear stability and variational principle

To deepen our understanding of stability, we have to examine the effect of larger perturbations.



a) Variational principle in 0-D

$$V(x) = -\int_x f(\xi)\mathrm{d}\xi$$ — pseudo-potential

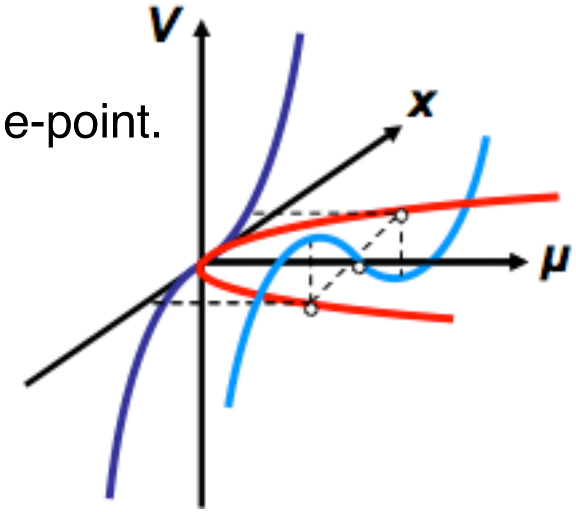
$$\dot{x} = f(x) = -V'(x)$$

$$\dot{x}^2 = -\frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = -\dot{V}$$

V will *decrease* along the ODE's trajectory as long as :

$$\dot{x} \neq 0 \Leftrightarrow V' \neq 0$$

$\dot{x} = 0$ if V reaches a minimum, a maximum, or a saddle-point.



Of course, only $V = min$ is stable — *nonlinearly*.

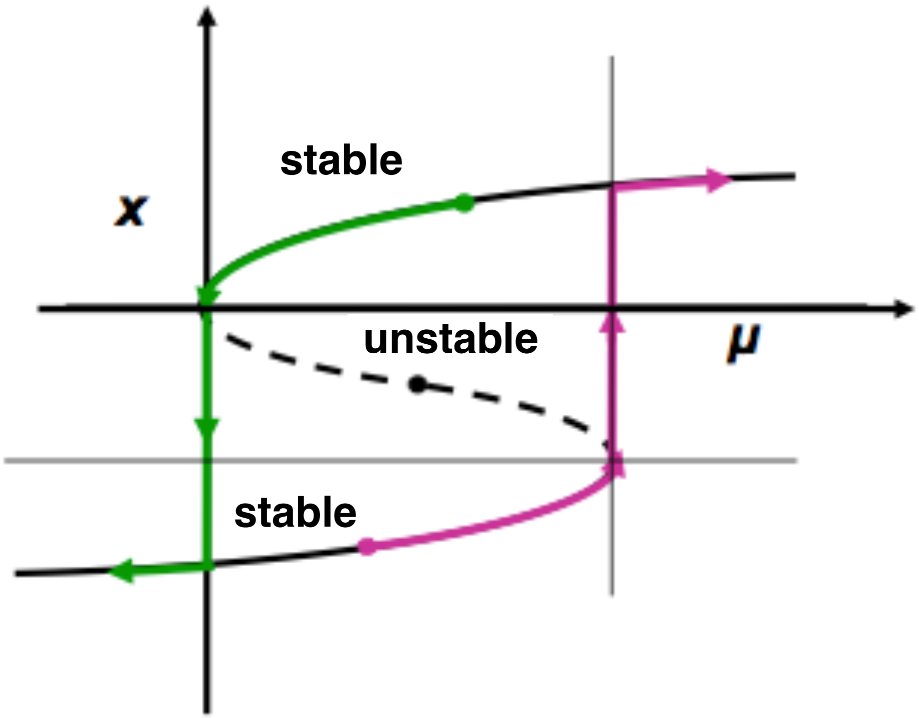
With this result, we turn back to the saddle-node bifurcation:

$$\dot{x} = \mu - x^2$$

$$V(x;\mu) = -\mu x + x^3/3 + c(\mu)$$

5. Bistability and hysteresis

The combination of two saddle-node bifurcations can create a hysteresis phenomenon (an *S*-shaped curve) :



$\dot{x} = \mu - x^2$: the top-left bifurcation

$\dot{x} = (\mu - 1) + (x + \frac{1}{2})^2$: the bottom-right bifurcation

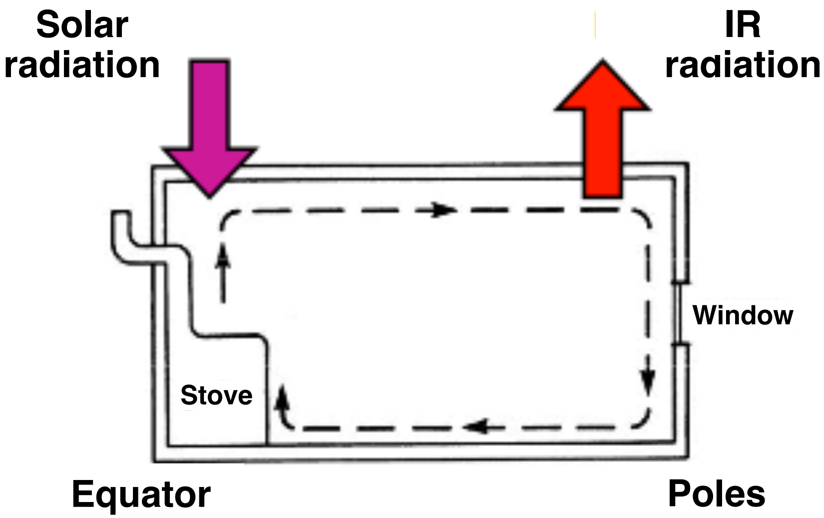
Outline, EBMs

Energy Balance Models (EBMs)

1. Radiation Balance
   – 0-D
   – 1-D in the meridional direction
2. EBMs, formulation and analysis
   – formulation in 0-D and 1-D
   – linear stability in 0-D and 1-D
3. Bifurcations in 0-D and 1-D
4. Nonlinear stability and variational principle
   – variational principle in 0-D
   – variational principle in 1-D
5. Comparison with 3-D GCM
6. Bistability and hysteresis

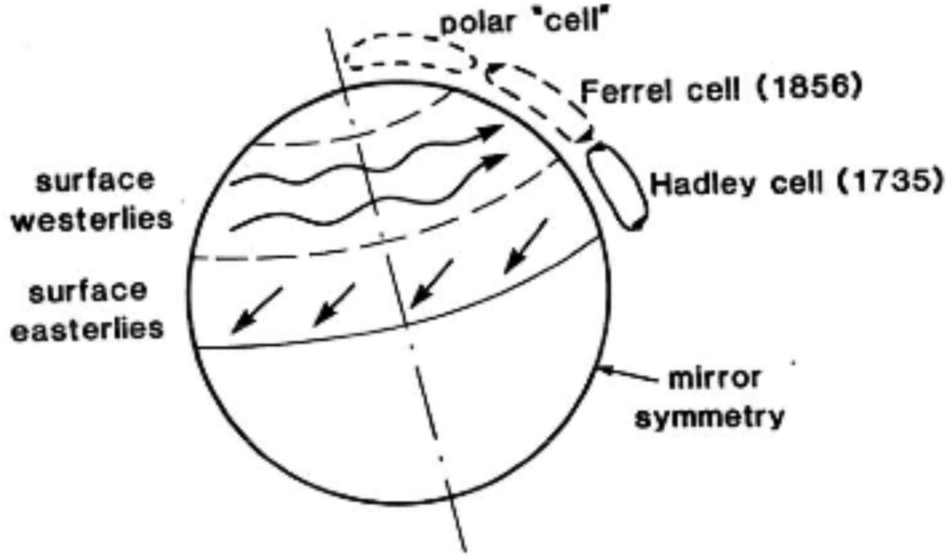
The mean atmospheric circulation

Direct Hadley circulation



Idealized view of the atmosphere’s global circulation.*

Observed circulation



Schematic diagram of the atmospheric global circulation.*

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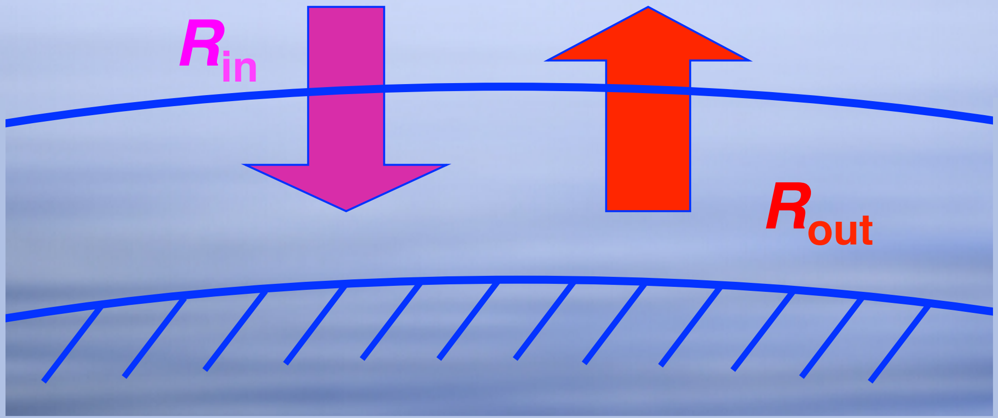
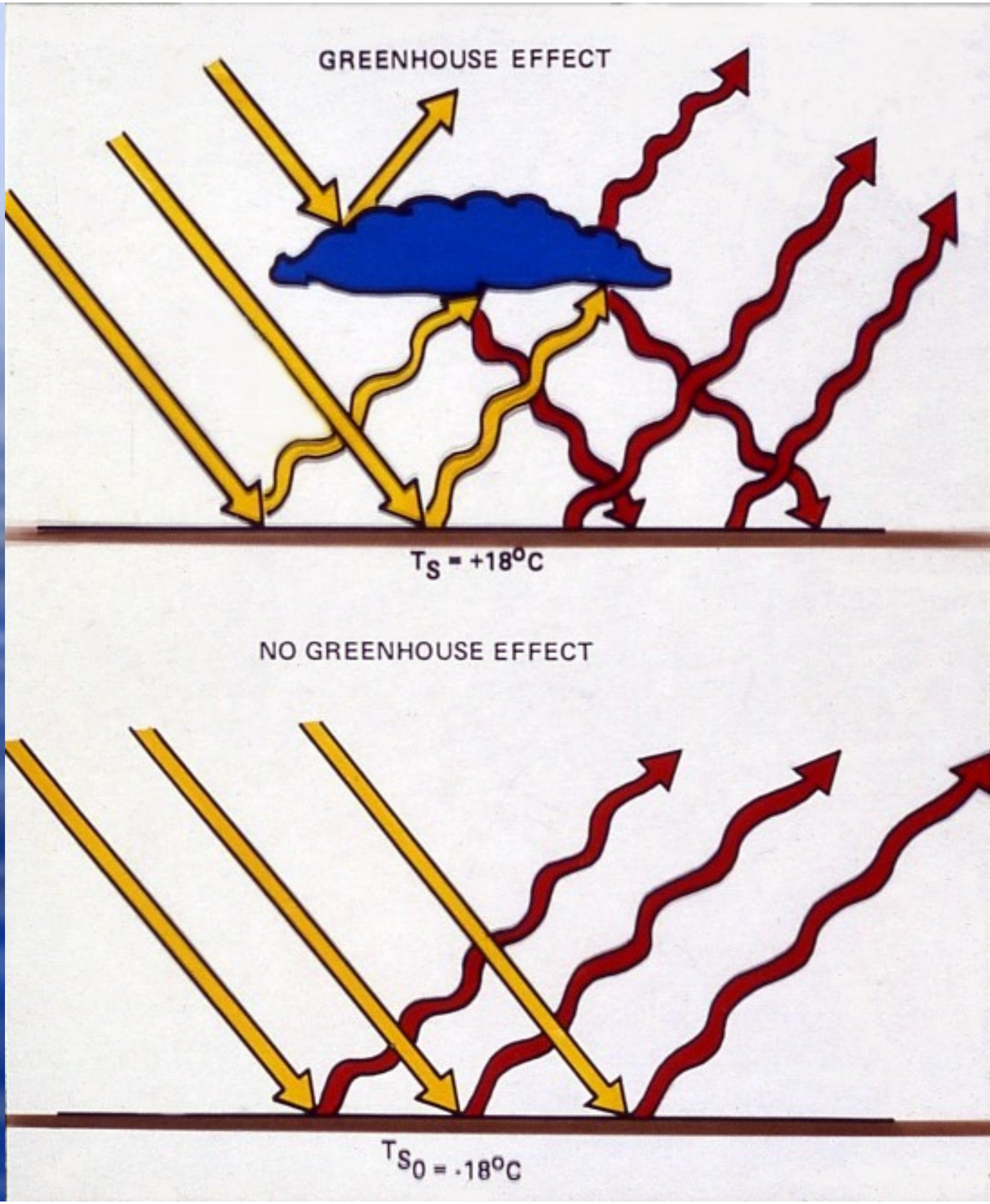
*From **Ghil and Childress (1987), Ch. 4**

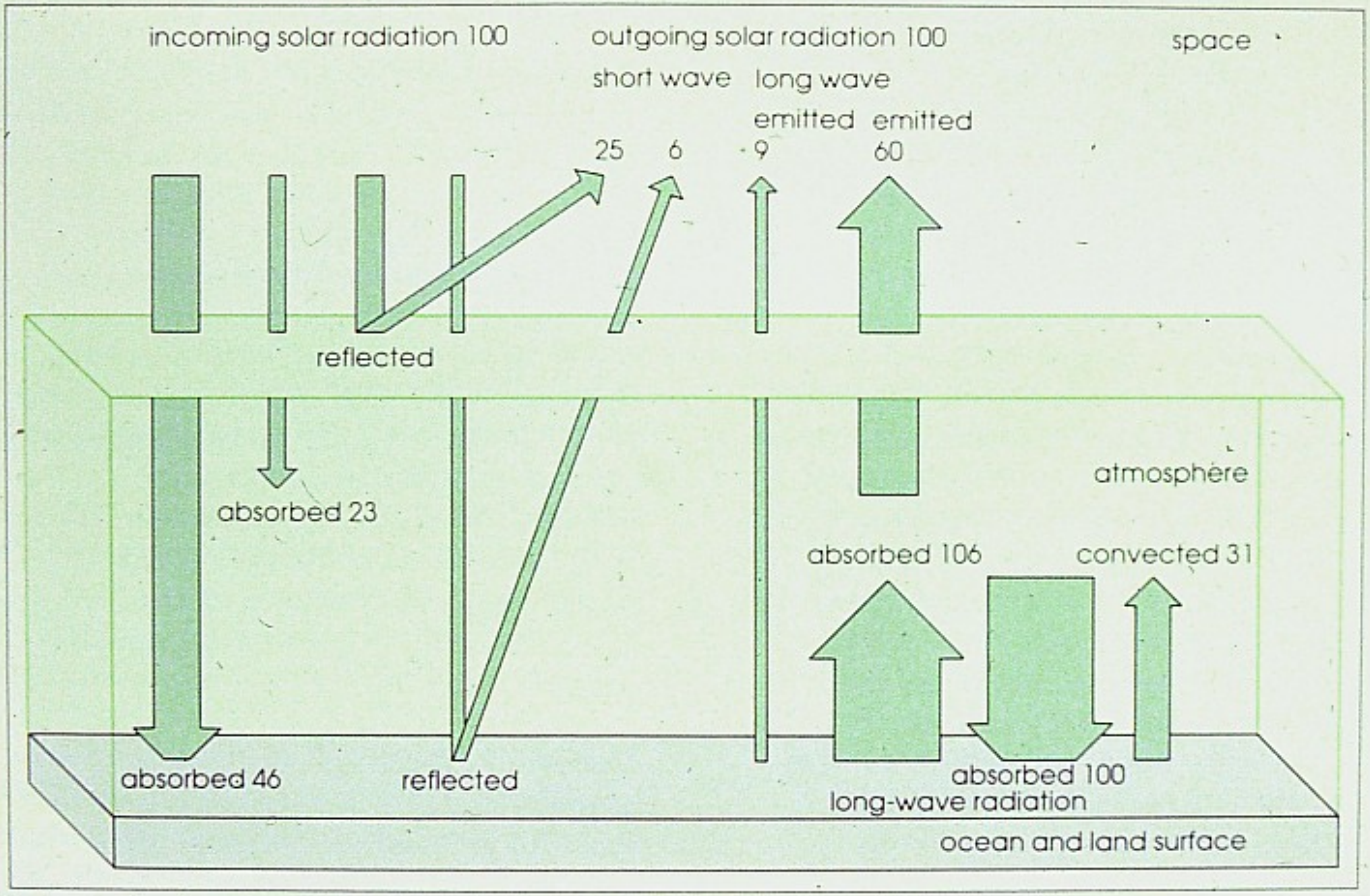
Radiative balance



Long-term equilibrium between incident (solar, ultra-violet + visible) radiation R_{in} and outgoing (terrestrial, infrared) radiation R_{out} dominates climate.

Refs. **[1] Egyptian scribe** (3000 B.C.) :
''The Sun heats the Earth,'' *Rosetta stone*, ll. 13–17.
[2] **Herodotus** (484 - cca. 425 B.C.)


GREENHOUSE EFFECT
Ts = +18°C
NO GREENHOUSE EFFECT
Ts0 = -18°C

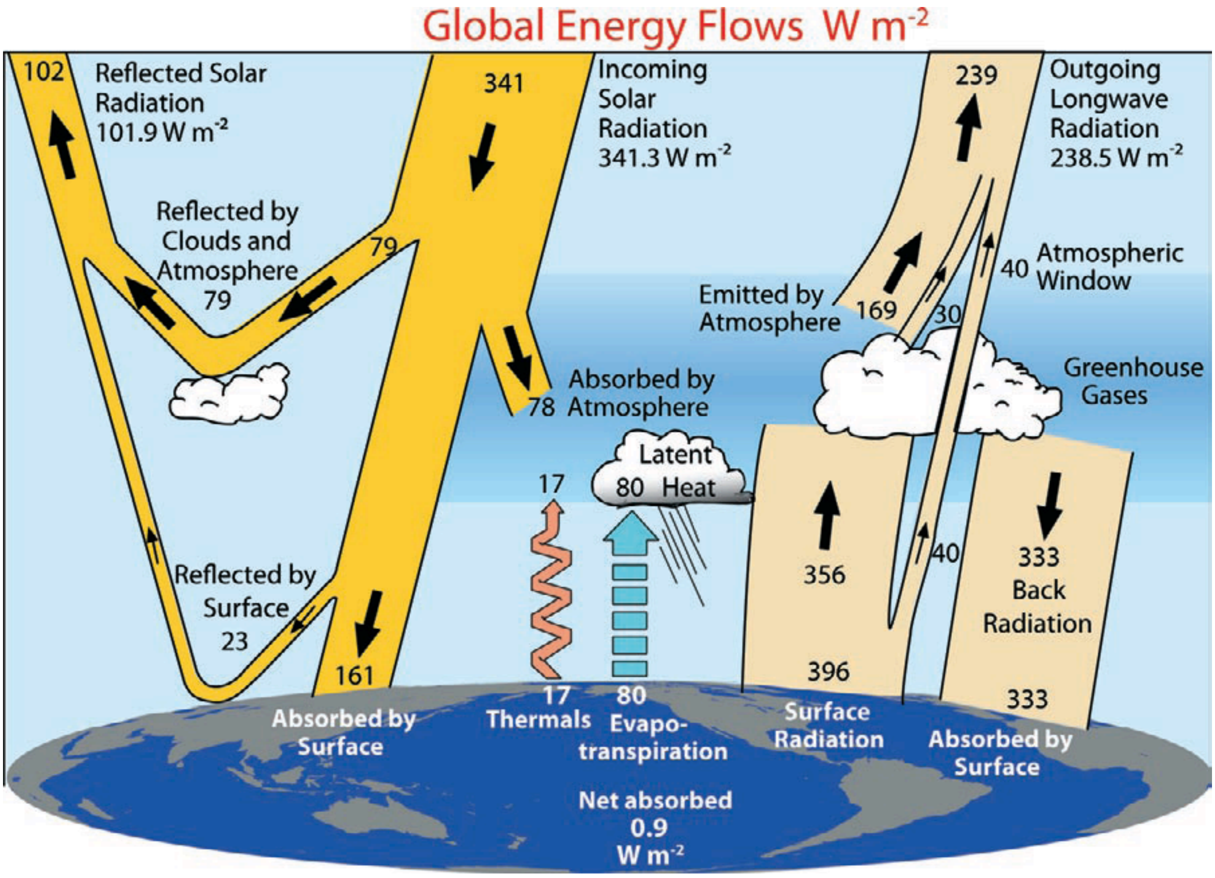


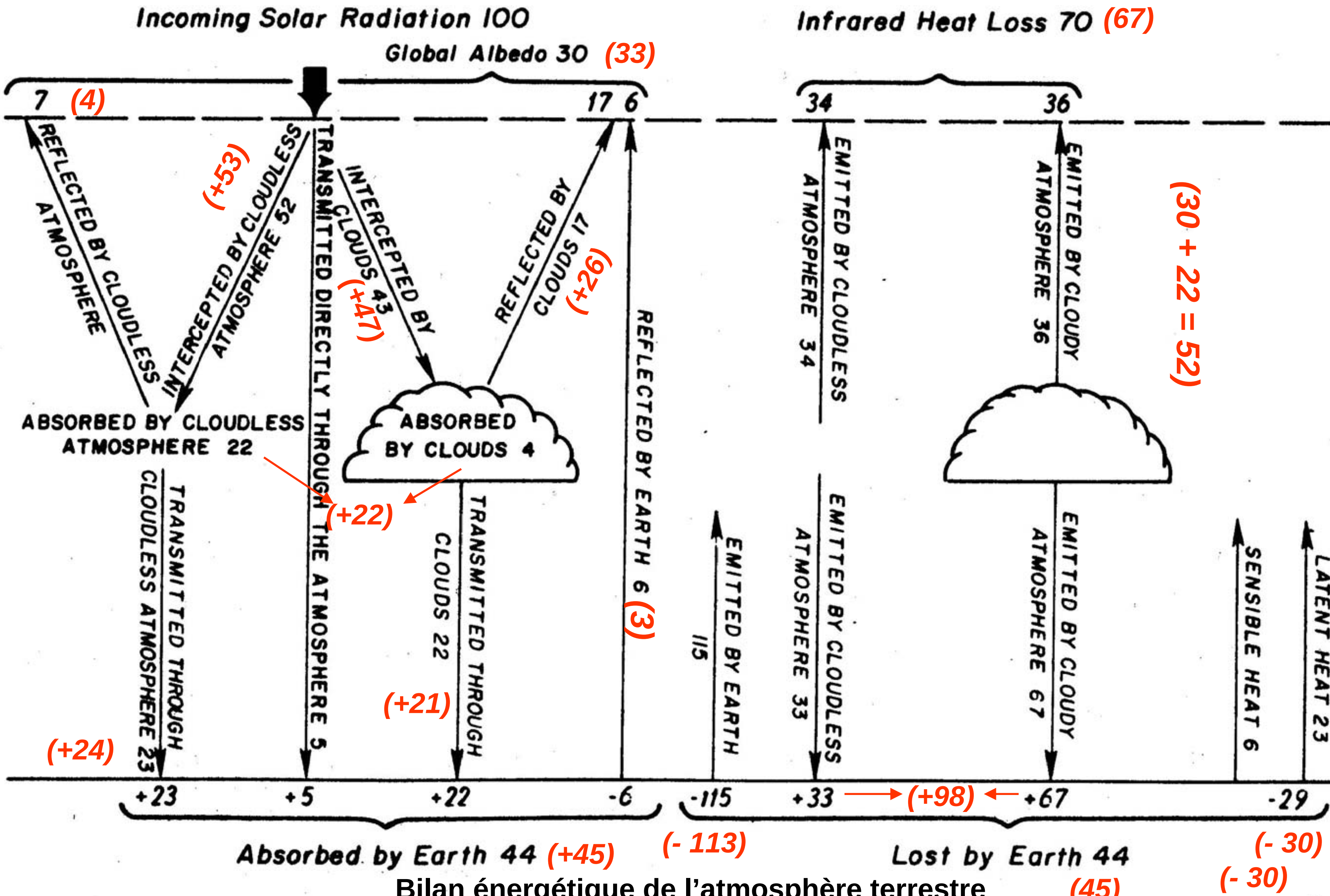
incoming solar radiation 100
outgoing solar radiation 100
space
short wave
long wave
emitted
emitted
25
6
9
60
reflected
absorbed 23
atmosphere
absorbed 106
convected 31
absorbed 46
reflected
absorbed 100
long-wave radiation
ocean and land surface

Earth's Global Energy Budget

K.E. Trenberth, J.T. Fasullo & J. Kiehl, 2009,

Bull. Amer. Meteorol. Soc., **90**(3), 311–323.





Bilan énergétique de l'atmosphère terrestre

Valeurs en rouge: cf. figure précédente.

D'après **Kuo-Nan Liou, 1980:** ***An Introduction to Atmospheric Radiation*** **(fig. 8.19)**

Energy balance models (EBMs)

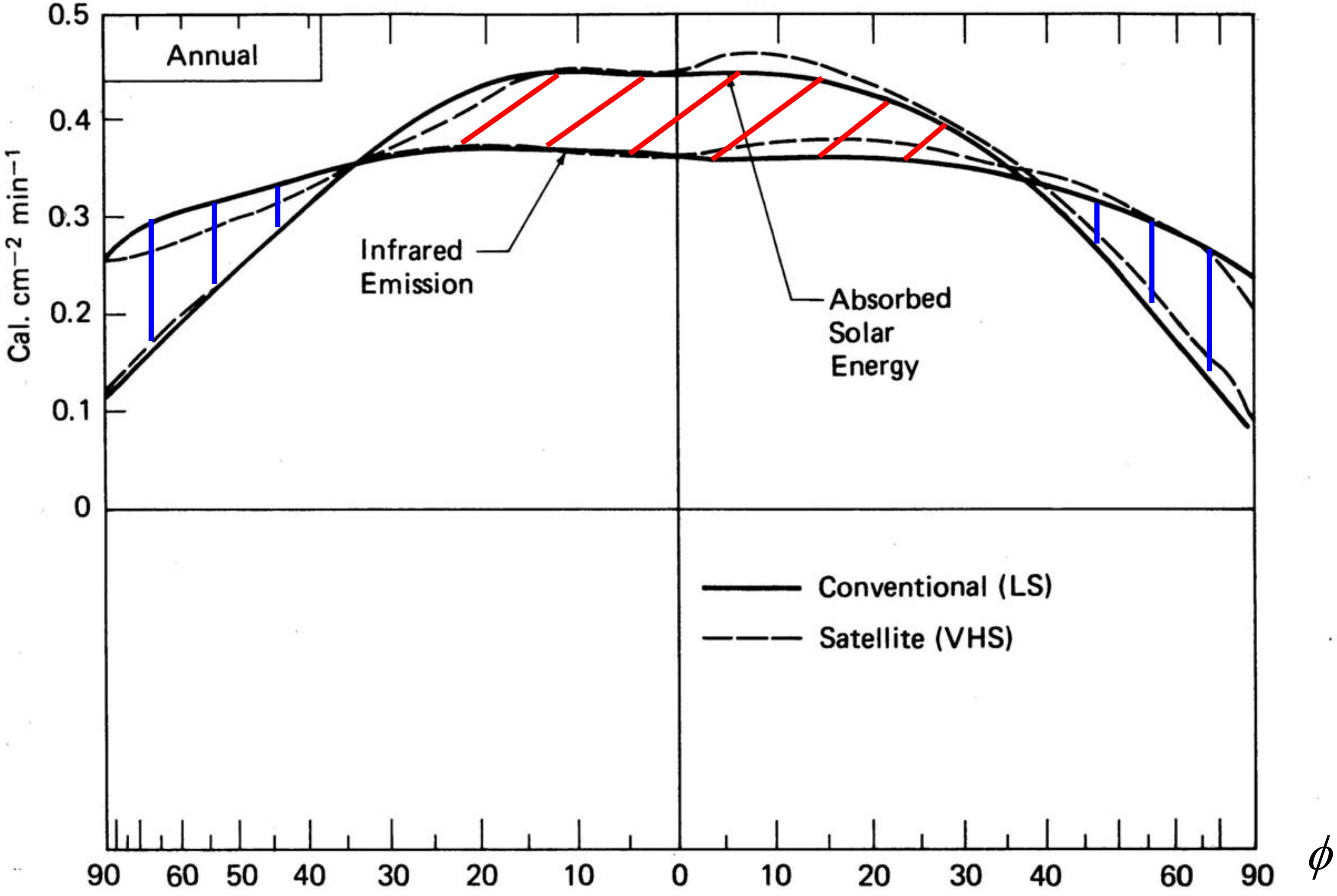
Problem 5: Compute the energy balance of Earth's atmosphere.

References

1. Reserve slides to this lecture.

2. Ghil, M., and S. Childress, 1987: Ch. 10 in *Topics in Geophysical Fluid Dynamics: Atmospheric Dynamics, Dynamo Theory and Climate Dynamics*, Springer-Verlag, New York, 485 pp.

3. Liou, K.-N., 2002: *An Introduction to Atmospheric Radiation*, 2nd ed., Academic Press, 583 pp. (compare also 1st ed., 1980)



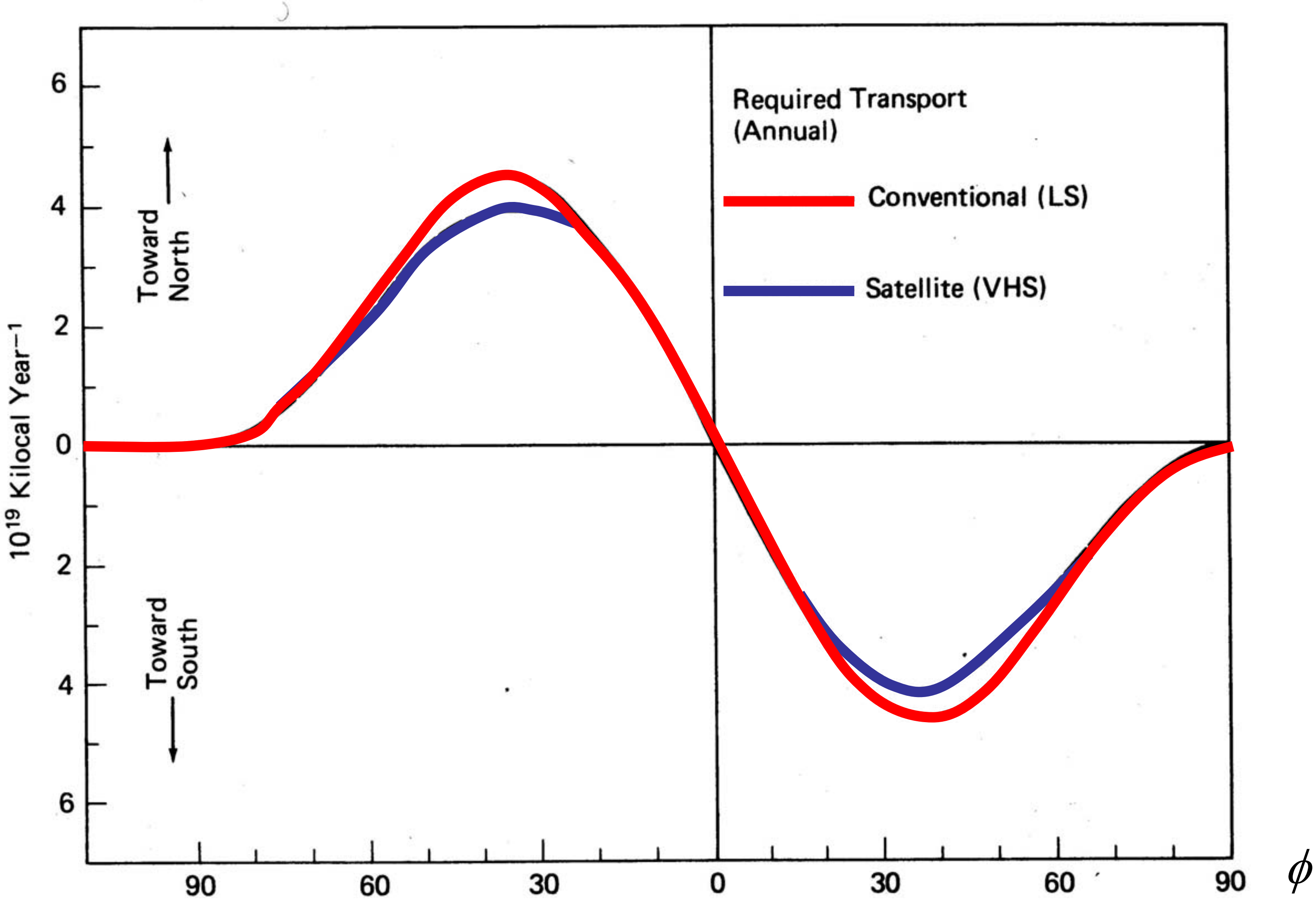
Bilan radiatif du système Terre-atmosphère en fonction de la latitude

$$R_i(\phi) - R_o(\phi) = \frac{1}{a.\cos(\phi)} \frac{\partial F}{\partial \phi}$$

Avec

a — le rayon de la Terre et

F — le flux atmosphérique et océanique de chaleur



Transport atmosphérique et océanique d'énergie en fonction de la latitude

$$F(\phi) = a\int_0^{\phi} \cos(\varphi)\left[R_i(\varphi) - R_o(\varphi)\right]d\varphi$$

Energy-balance models (EBMs)

$$C\frac{\partial T}{\partial t} = R_i - R_o + D$$

C — local calorific capacity

T — local surface temperature

R_i — incident solar radiation

R_o — terrestrial radiation towards space

D — heat redistribution ('diffusion')

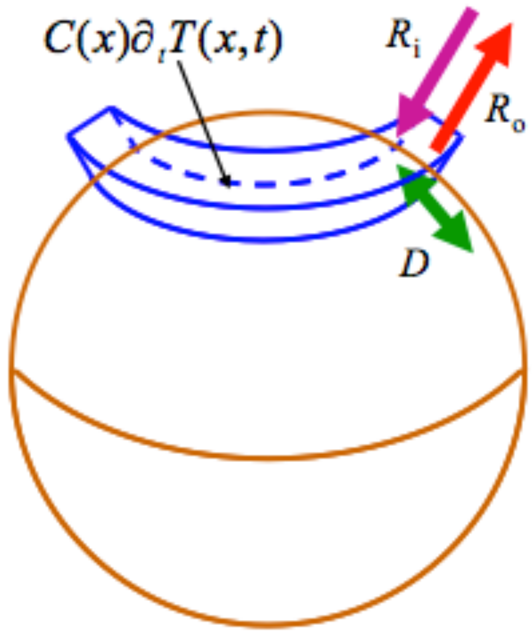
Comments:

1. C, R_i, R_o and D have to be calculated ("parameterized") according to $T = T(x,t)$

2. The model's main characteristic is R_i

$$R_i = Q(x)\{1-\alpha(x,T)\}$$

with α the local albedo.



0-D version (averaged over the globe)

$$C\frac{d\bar{T}}{dt} = R_i - R_o = Q\left\{1 - \alpha(\bar{T})\right\} - \sigma\bar{T}^4 m(\bar{T})$$

$\bar{T}$ — average surface temperature

t — time (in thousands of years)

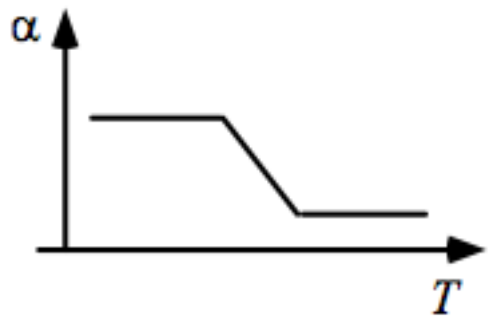
Q — incident solar flux

α — albedo

C — calorific capacity

σ — Stefan–Boltzmann constant

m — greenhouse effect factor



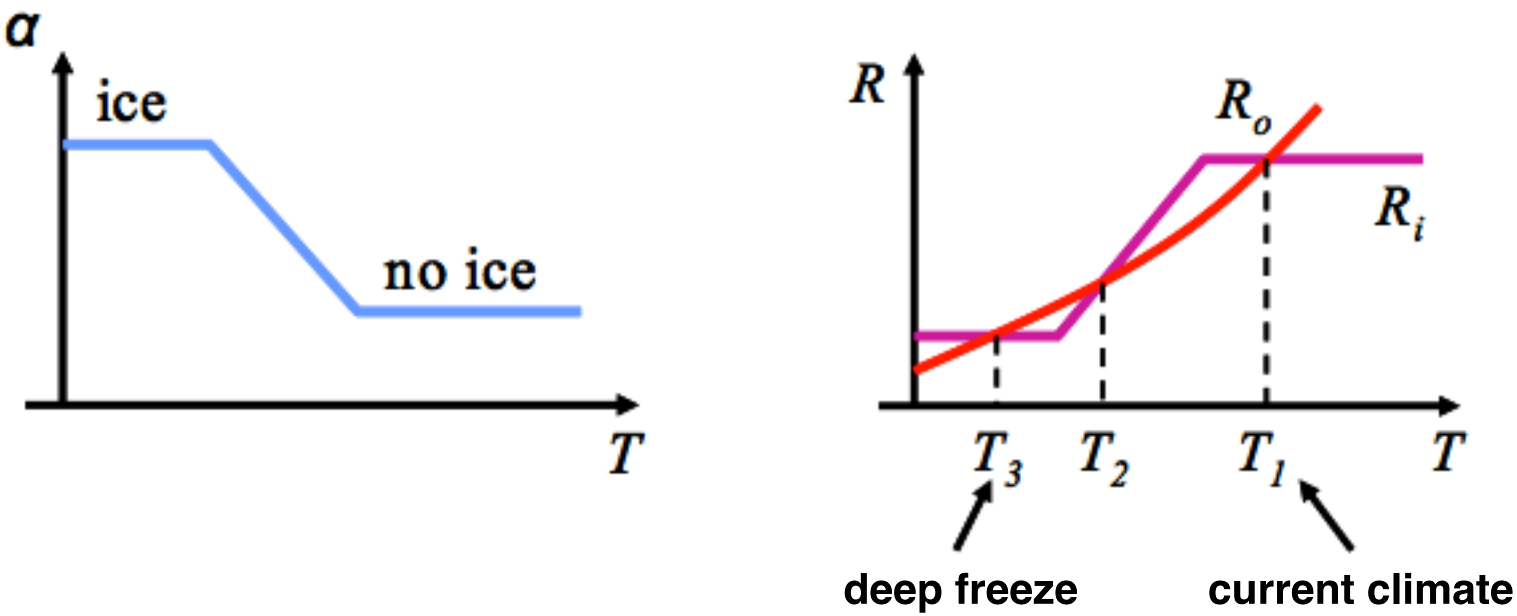
Comments:

α depends on the ice and snow cover, on cloud cover, etc. (implicit variables). All is parameterized as a function of the explicit variable $\bar{T}$.

0-D EBM, I: Model solutions

We want to write T as: $T = T(t; T_0, Q, c, ...)$

Stationary solutions: $Q\{1 - \alpha(T)\} - \sigma T^4 = 0$



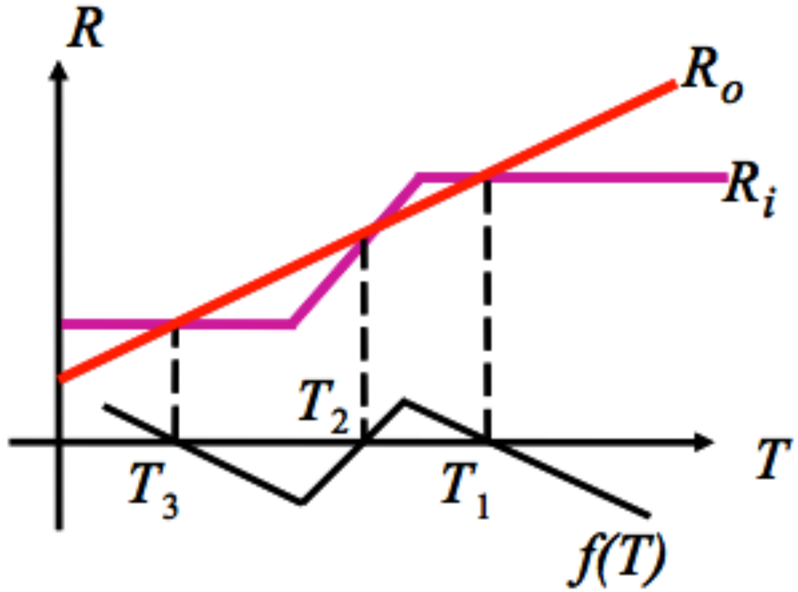
What happens if the sun "blinks" and $T = T_1 + \Delta T$?

We have to go back to the original equation, which depends on time.

0-D EBM, II: Stability condition

$$C\partial_t T = R_i - R_o = f(T)$$

$$R_i = Q\{1 - \alpha(T)\}$$
$$R_o = A + BT$$



We set $T = T_j + \theta$:

$f(T_j) = 0,$
$f(T) = f(T_j) + f'(T_j)\theta + ...$

Let's define $\lambda_j \equiv f'(T_j)/c$

$$\partial_t \theta = \lambda_j \theta \Rightarrow \theta = e^{\lambda_j t}\theta_0$$

If $\lambda_j < 0$ stable;

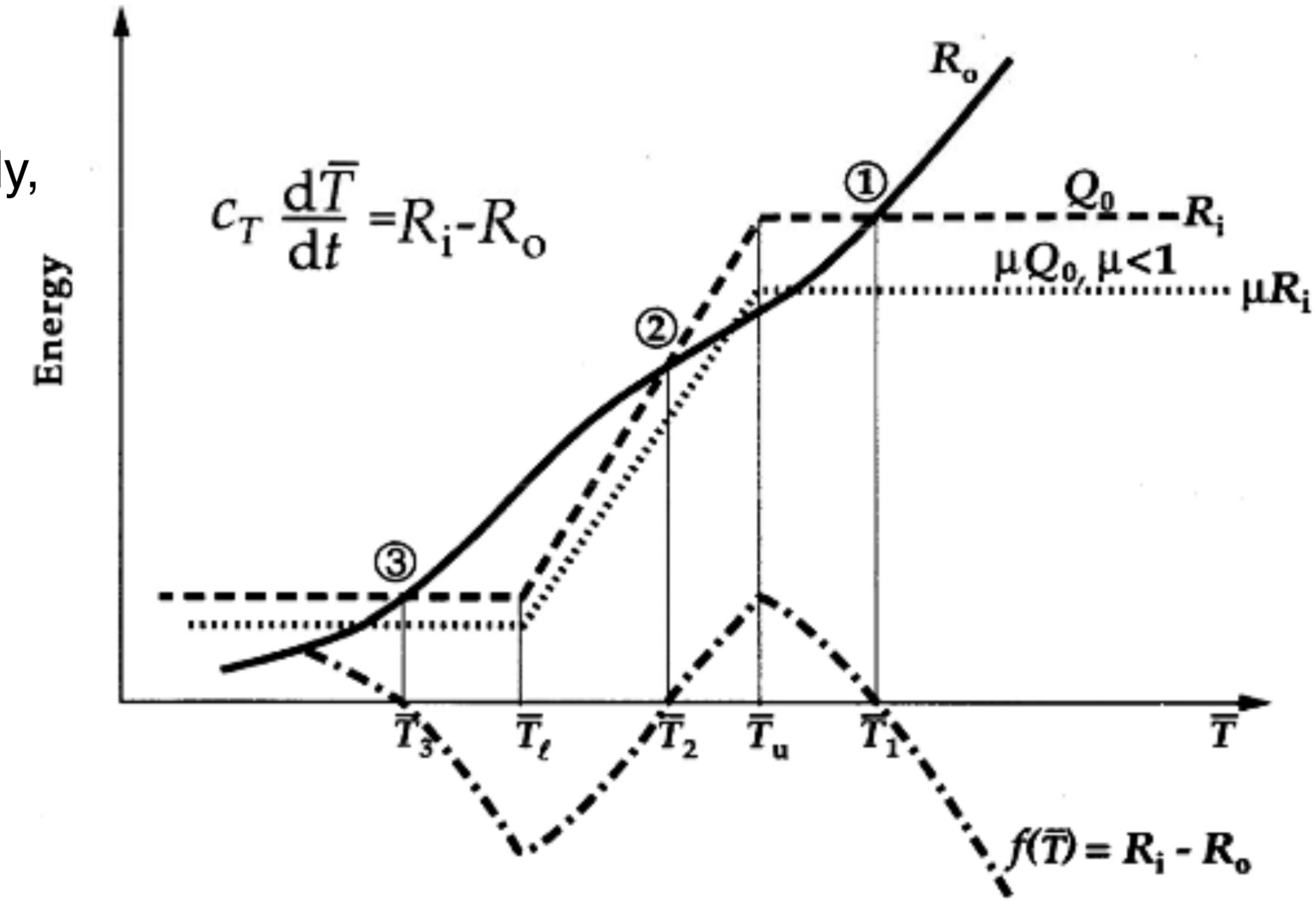
if $\lambda_j > 0$ unstable.

Comment: in the 1-D case $\lambda_j \to \lambda_j^{(0)}$;
$\lambda_j \sim 1/c$

0-D EBM, III: Changes in parameters

What happens if the insolation parameter μ changes, i.e., the "solar constant" changes? This may represent a change in solar luminosity, orbital parameters or in the optical properties of the atmosphere.

❖The model's three "climates" shift in value and, possibly, in number.



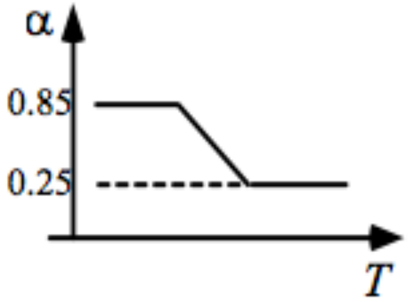
1-D version ('classic' EBM)

$$C(x)T_t = R_i - R_o + D$$

T — temperature
x — latitudinal coordinate
$\tilde{T}(x)$ — the observed climate

Boundary conditions: $T_x(0) = T_x(1) = 0$
$x = 0$ Pole (North)
$x = 1$ Equator

$$R_i = Q(x)\{1-\alpha\}$$
$$= Q(x)\{1 - b(x) + c_1 T\}_c$$
$$R_o = \sigma T^4\{1 - m \tanh(c_3 T^6)\}$$



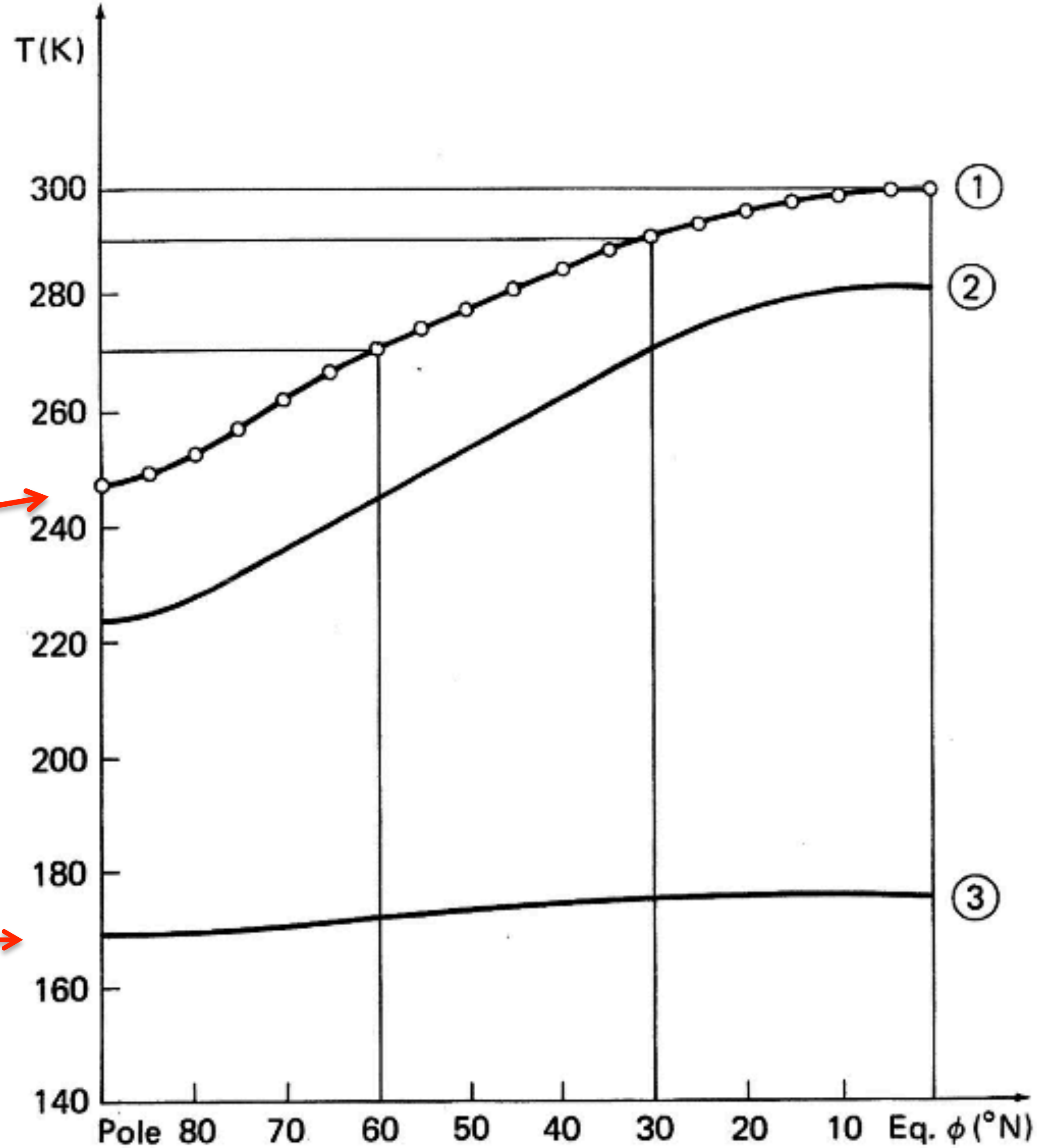
$$D = \frac{1}{\sin\frac{\pi x}{2}} \partial_x \sin\frac{\pi x}{2}\{k(x) + k_s(x) g(\tilde{T})\} T_x$$

Questions: 1. Stationary solutions ('climates')?
2. Stability?
3. Perturbation & bifurcation? $Q \to \mu Q \quad (\mu = 1)$

The three climates of the 1-D model

Present climate

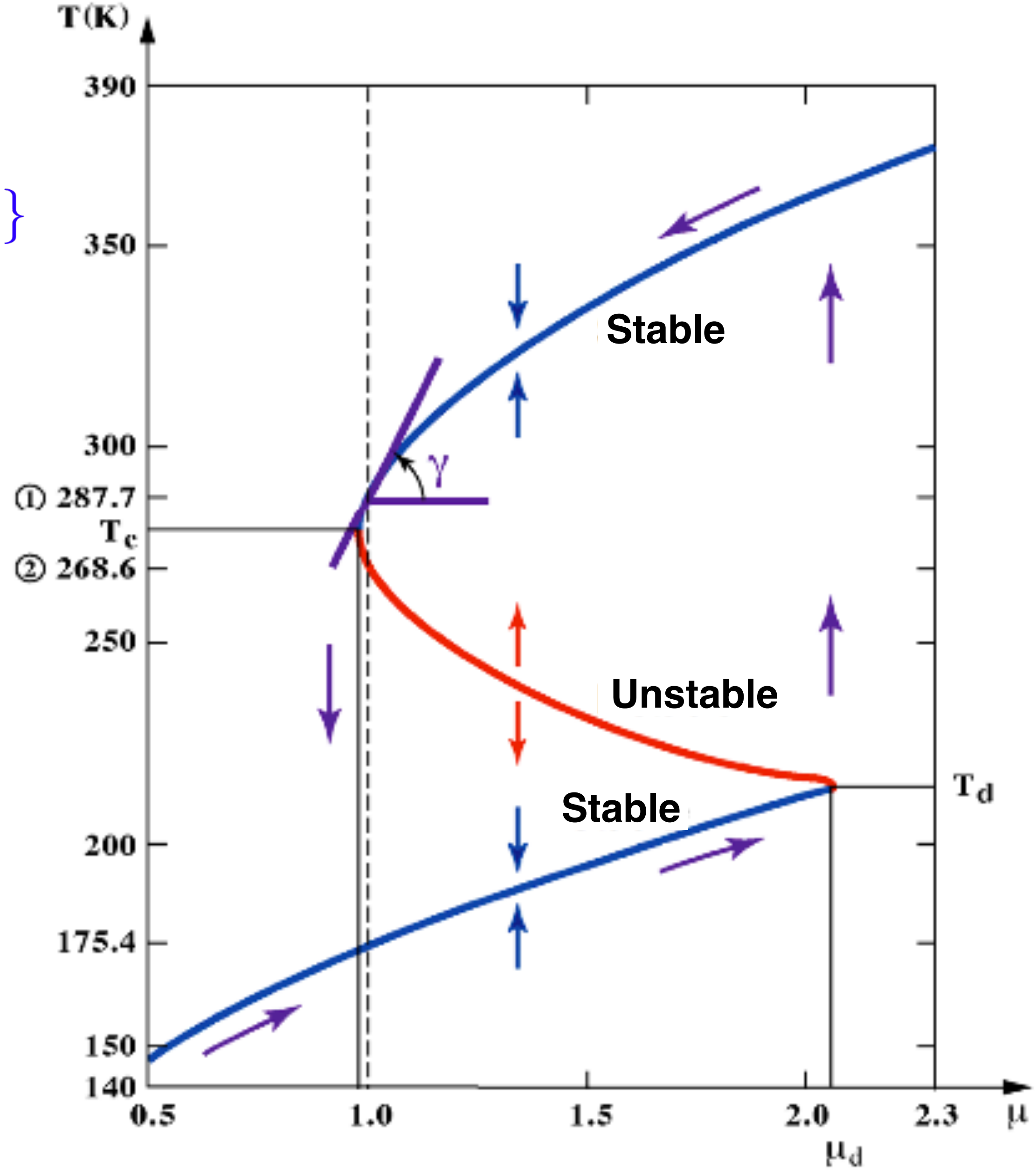
"Deep freeze" or snow-ball Earth



1-D EBM: Bifurcation diagram

$$
\begin{aligned}
C(x)T_t = &\{k(x,T)T_x\}_x \\
&+\mu Q_0\{1-\alpha(x,T)\} \\
&-g(T)\sigma T^4
\end{aligned}
$$

$T_x = 0$ at $x = 0, 1$



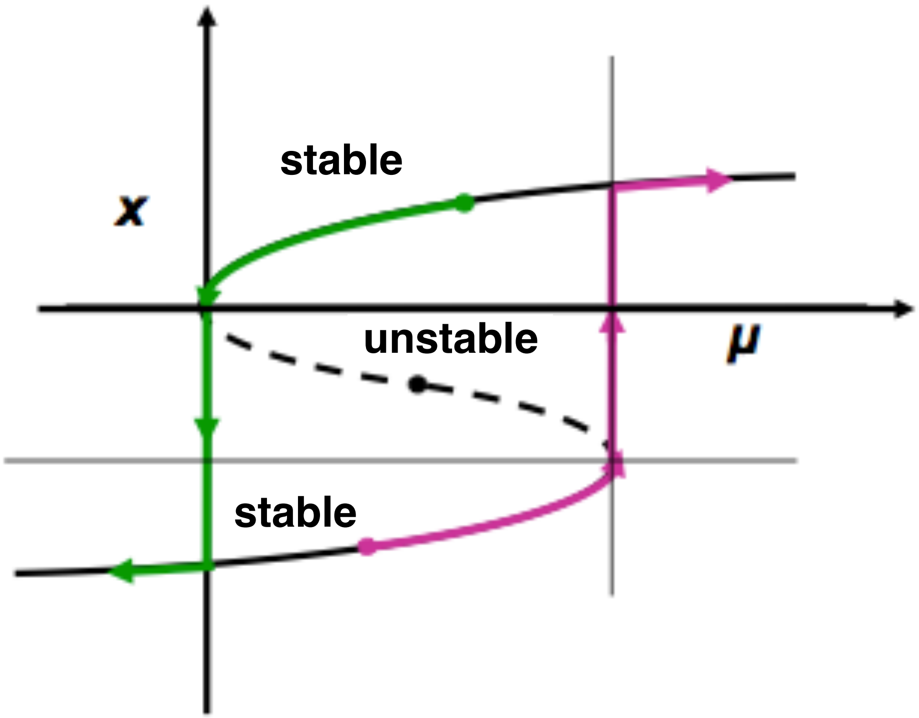
Climate sensitivity:

$$\gamma = \frac{dT}{d\mu} \cong 0.01$$

(1K per % of Q)

5. Bistability and hysteresis

The combination of two saddle-node bifurcations can create a hysteresis phenomenon (an *S*-shaped curve) :



$\dot{x} = \mu - x^2$: the top-left bifurcation

$\dot{x} = (\mu - 1) + (x + \frac{1}{2})^2$: the bottom-right bifurcation

Elementary bifurcation problems for PDEs

Problem 6: Compute the saddle-node bifurcation for the reaction-diffusion problem

$$u_t = k u_{xx} + \mu(1 - u^2)$$

with suitable boundary conditions on the interval

$$0 \leq x \leq 1.$$

Energy Balance Models (EBMs)

Budyko, Sellers and Held-Suarez-North

10.2. Energy-Balance Models (EBMs): Multiple Equilibria 311

Table 10.1. Comparison of Budyko's and Sellers' models.

Heat Flux	Budyko	Sellers
$R_i = Q(1 - \alpha(T))$ Absorbed solar radiation, as a function of ice-albedo feedback	Step-function albedo $\alpha = \begin{cases} \alpha_M, & T < T_s, \\ \alpha_m, & T \geq T_s, \end{cases}$ $\alpha_M > \alpha_m$, $T_\ell \leq T_s \leq T_u$	Ramp-function albedo $\alpha = \begin{cases} \alpha_M, & T < T_\ell \\ \alpha_M - \frac{T - T_\ell}{T_u - T_\ell}(\alpha_M - \alpha_m) & T_\ell \leq T < T_u, \\ \alpha_m, & T_u \leq T \end{cases}$,
R_0 Outgoing IR radiation	Linear, empirical $A + BT$	Stefan-Boltzmann law with greenhouse effect $\sigma T^4\{1 - m \tanh(T^6/T_0^6)\}$
$\nabla \cdot F$ Horizontal flux divergence	Newtonian cooling $\kappa\{T(\phi) - \overline{T}\}$	Eddy-diffusive $\nabla \cdot \{k(\phi)\nabla T(\phi)\}$

2nd column:
Budyko (1968, 1969)

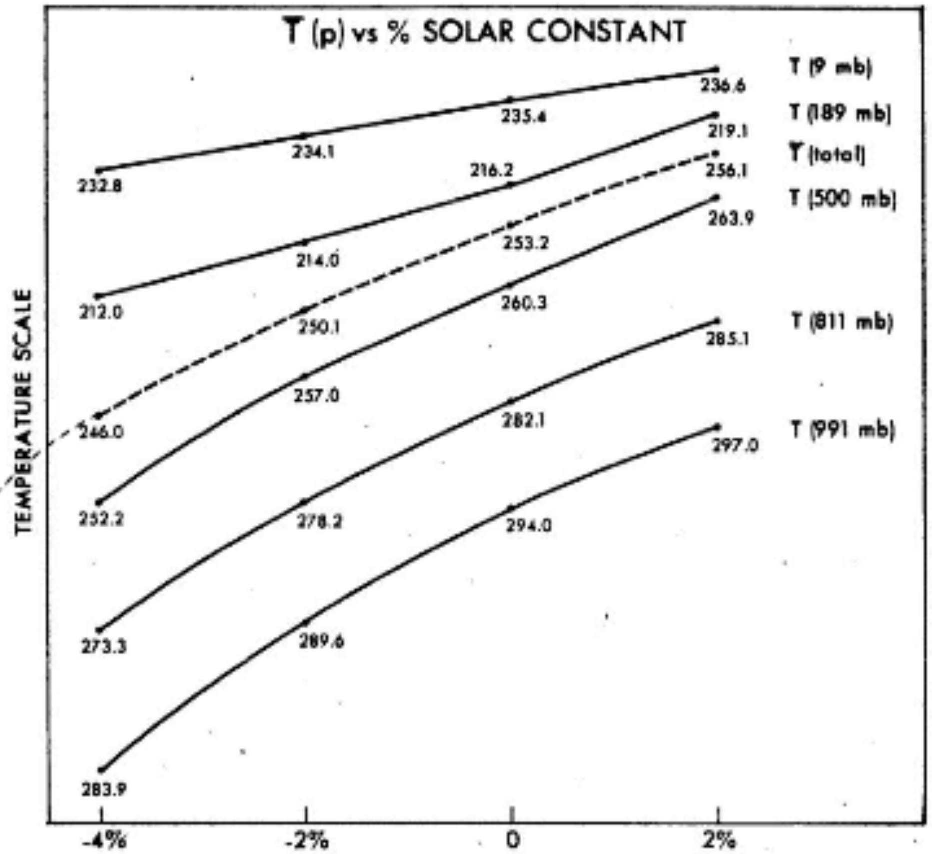
3rd column:
Sellers (1969)

In red:
the "mixed" version of
Held & Suarez (1974)
and North (1975a, b)

Climate sensitivity to insolation in a General Circulation Model (GCM)

"As stated in the Introduction, it is not, however, reasonable to conclude that the present results are more reliable than the results from the one-dimensional studies mentioned above simply because our model treats the effect of transport explicitly rather than by parameterization."*

"Nevertheless, it seems to be significant that both the one-dimensional and three- dimensional models yield qualitatively similar results in many respects."*



Area-mean temperatures for various model levels, as well as a mass-weighted mean temperature for the total model atmosphere. Based on 4 GCM runs: control, –4%, –2% and +4%. Units are in degrees K.

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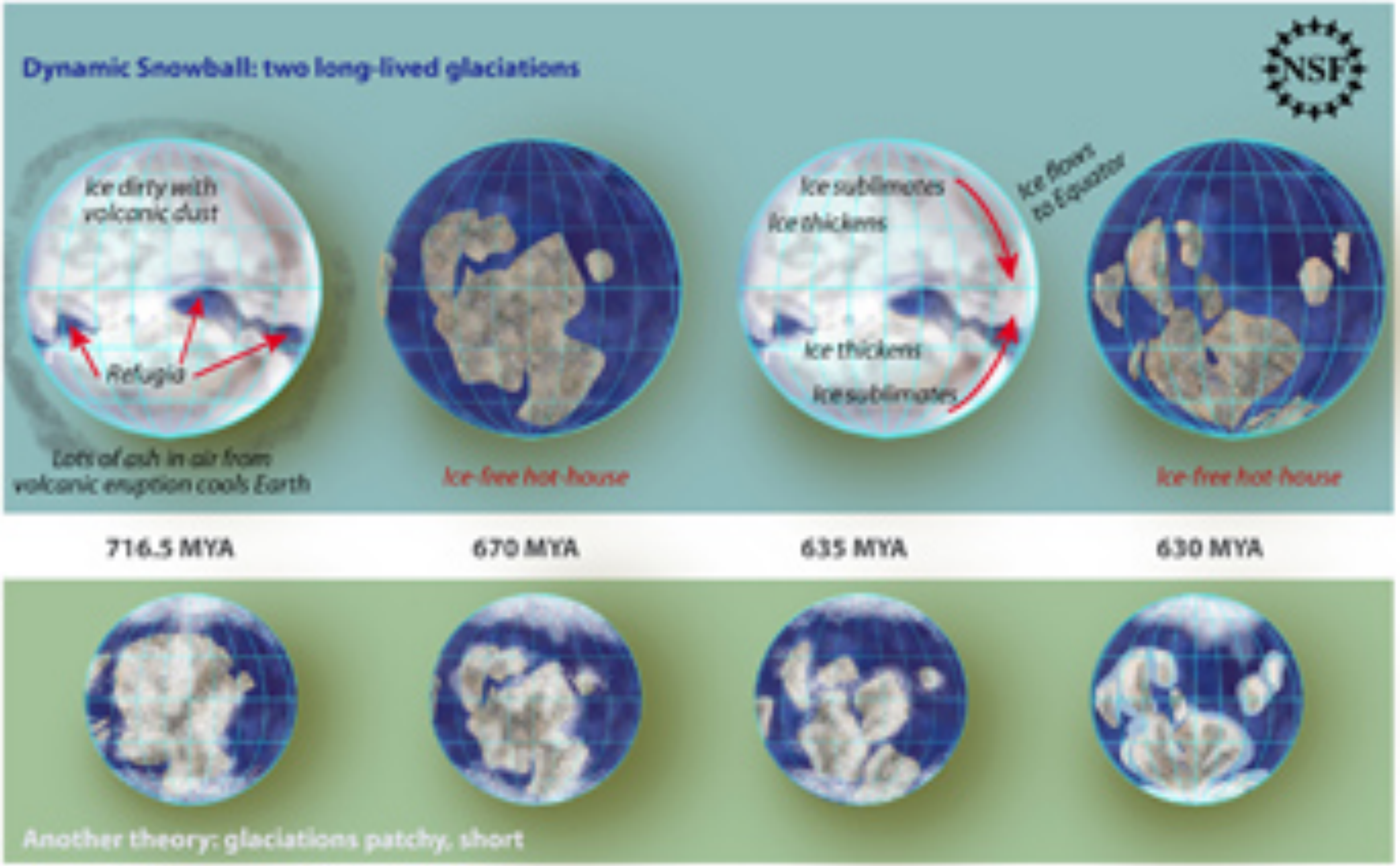
* From Wetherald and Manabe, 1975, *J. Atmos. Sci.*, **32,** 2044–2059.

Snowball Earth — Erstwhile a “theory”; now a “fact”?



https://www.nsf.gov/news/news_images.jsp?cntn_id=116410&org=NSF

Double-well potential in 2-D

1-D EBM of Budyko-Sellers-North, cf. Held & Suarez (*Tellus,* 1974); North *et al.* (*JAS*, 1979).

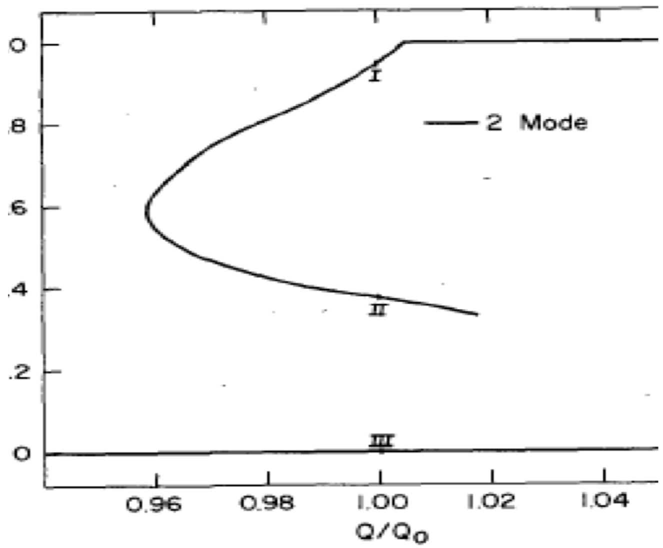
Taking $x = \sin(latitude)$ and $k(x, T) = k_0$,
We get the semi-linear parabolic PDE

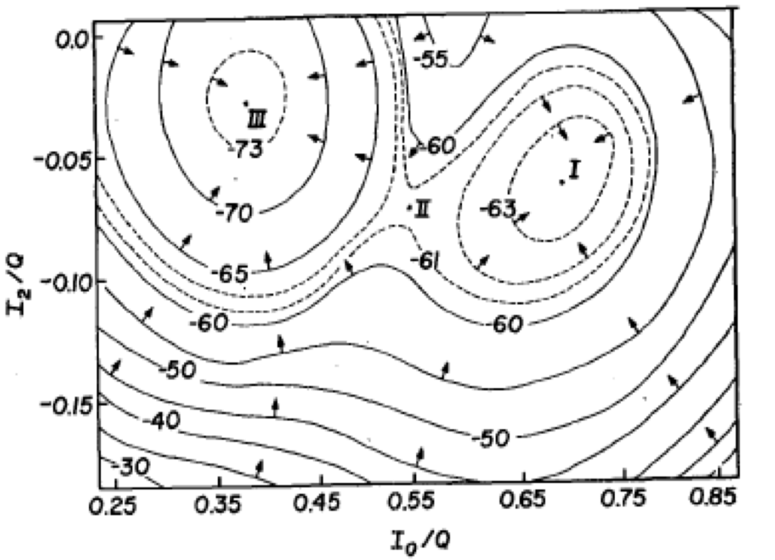
$$CT_t = [k_0(1 - x^2)T_x]_x + Q(x)[1 - \alpha(T)] - I(T)$$

which yields the variational principle:

$$F\{T(x)\} = \int \{\frac{1}{2}k_0(1 - x^2)T_x^2 - Q(x)A(T) + R(T)\}\, dx,$$ where

$$A(T) = \int^T [1 - \alpha(T)]\, dT,$$ and $$R(T) = \int^T I(T)\, dT.$$



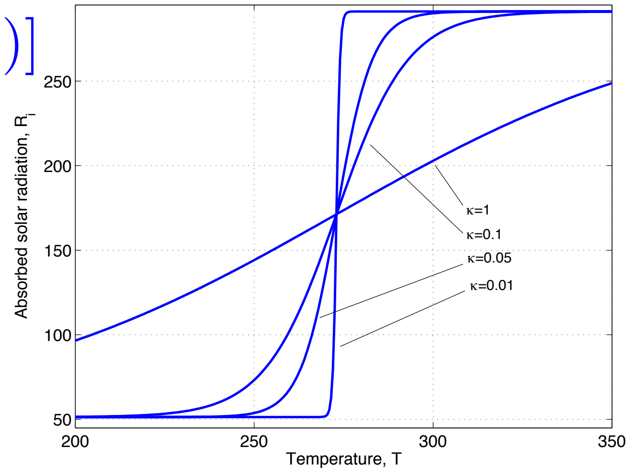


Distance to "tipping points"?

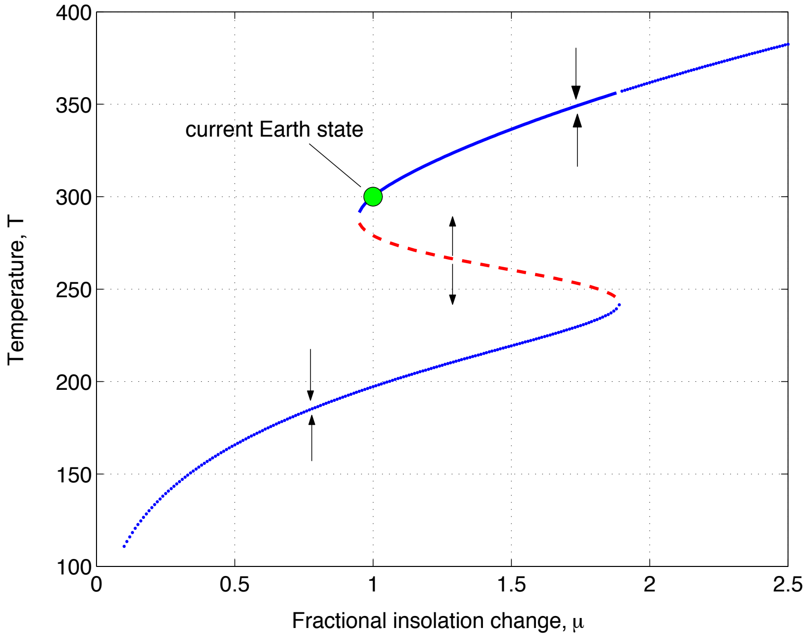
Slightly modified 0-D EBM (Zaliapin & Ghil, *NPG, 2010)*

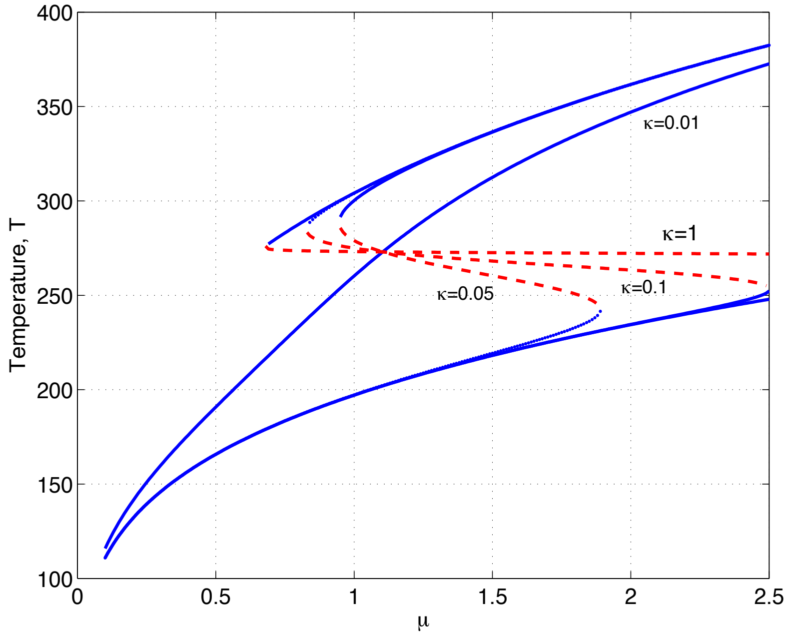
$$c\dot{T} = \mu Q_0(1-\alpha(T)) - \sigma T^4[1 - m\tanh((T/T_0)^6)]$$

$$\alpha(T;\kappa) = c_1 + c_2\frac{1-\tanh[\kappa(T-T_c)]}{2}$$



T_c is the ice-margin temperature,
while κ is an extra "Budyko-vs.-Sellers" parameter





Concluding remarks, I

- Tipping points and bifurcations: multiple equilibria and rapid transitions between them.
- Prediction of the transitions? To follow.
- Transitions between more general types of behavior — limit cycles, strange attractors — likewise to follow.

Some general references

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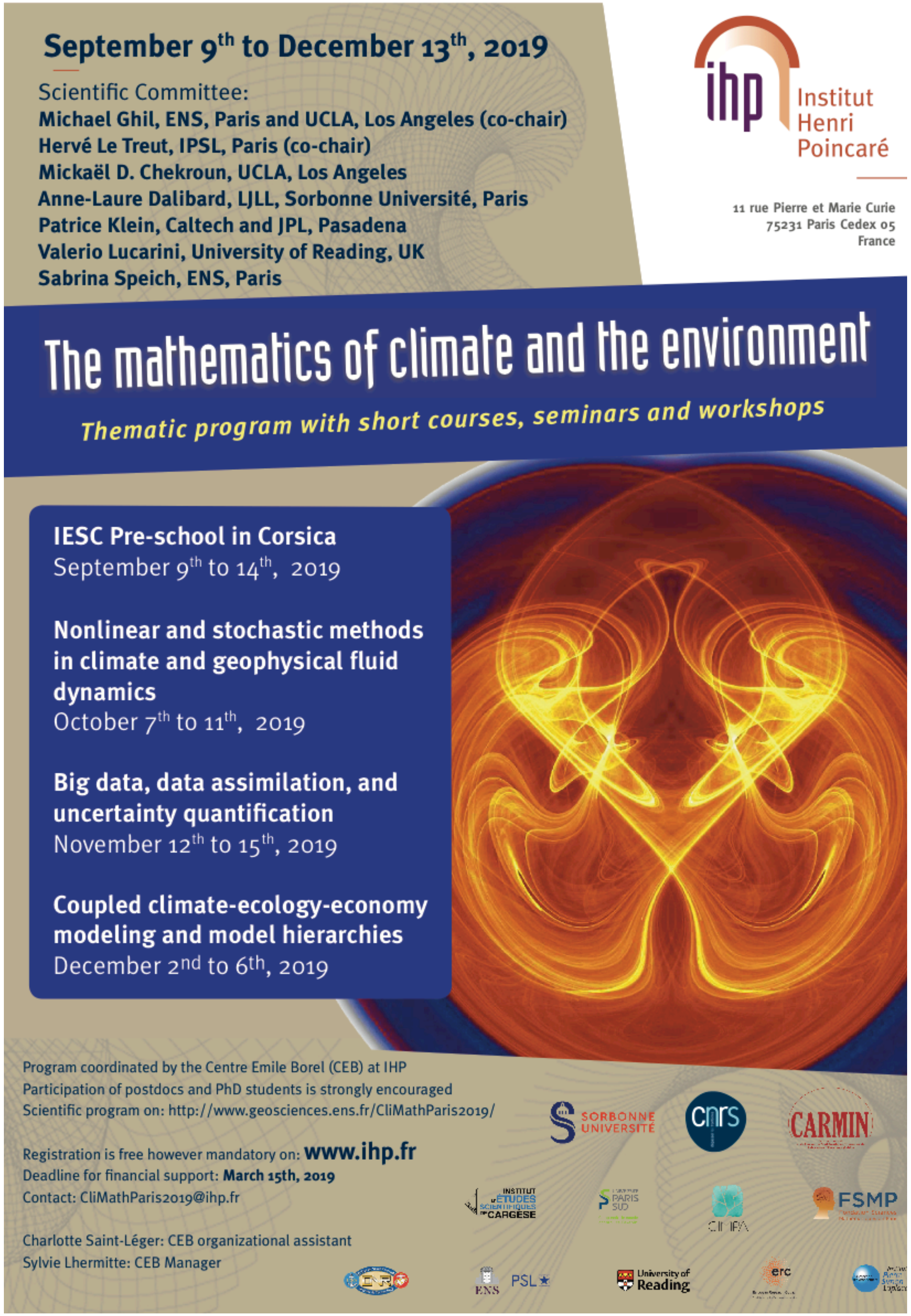

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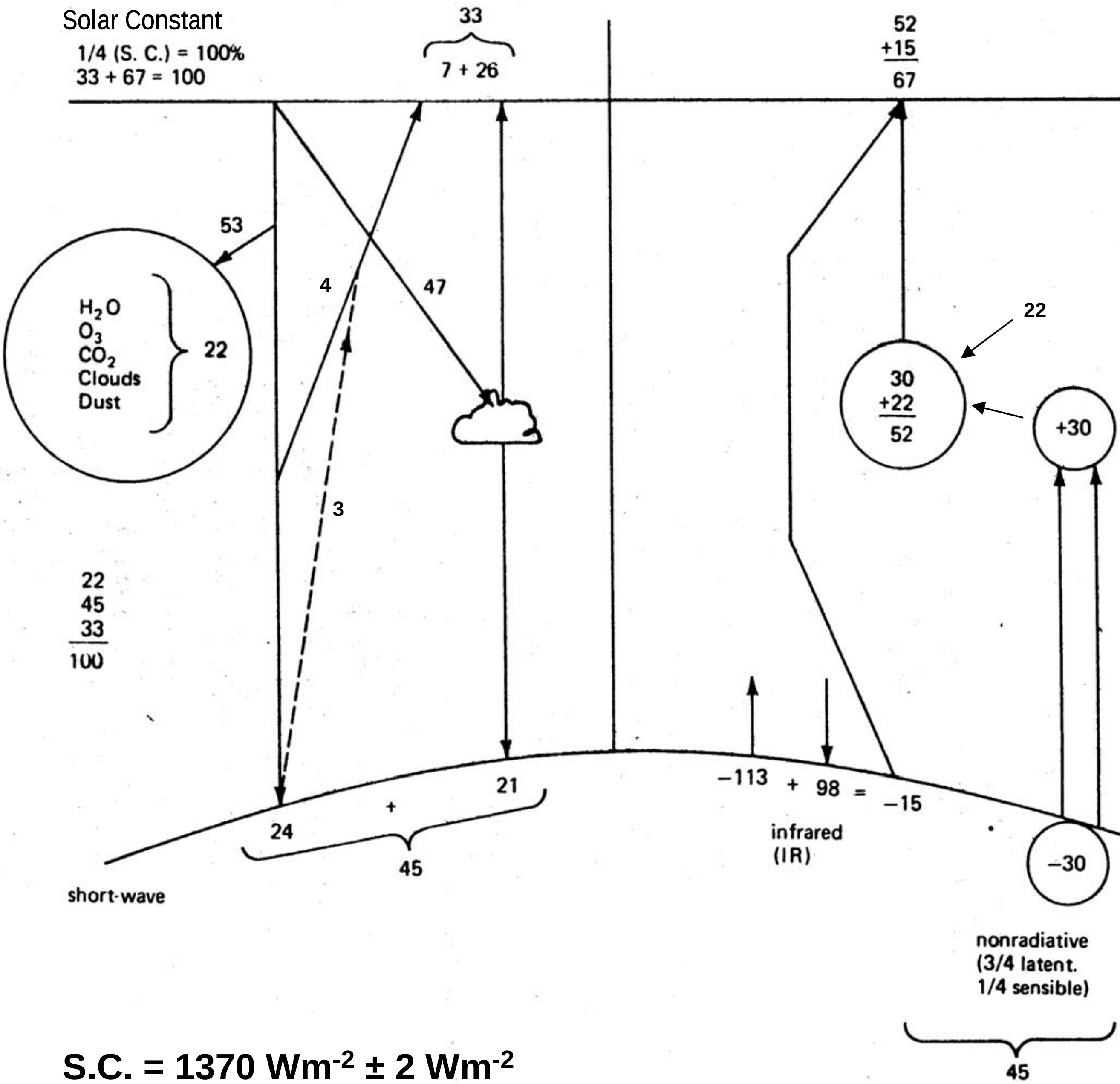

S.C. = 1370 Wm^{-2} ± 2 Wm^{-2}