Mathematical Problems in Climate Dynamics, CIMA + IFAECI

Univ. of Buenos Aires, 2–13 November 2018

Basic Facts of GFD + Atmospheric LFV, Wind-driven Oceans, Paleoclimate & "Tipping Points"

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Please visit these sites for more info.

https://dept.atmos.ucla.edu/tcd, http://www.environnement.ens.fr/
and https://www.researchgate.net/profile/Michael_Ghil

Overall Outline

- **Lecture I: Observations and planetary flow theory (GFD[⌘])**
- **Lecture II: Atmospheric LFV[*] & LRF[**]**
- **Lecture III: EBMs[+], paleoclimate & "tipping points"**
- **Lecture IV: Nonlinear & stochastic models—RDS[❖]**
- **Lecture V: Advanced spectral methods–SSA[±] *et al.***
- ⇨ **Lecture VI: The wind-driven ocean circulation**

(⌘) GFD = Geophysical fluid dynamics
(*) LFV = Low-frequency variability
(**) LRF = Long-range forecasting
(+) EBM = Energy balance model
(❖) RDS = Random dynamical system
(±) SSA = Singular-spectrum analysis

Motivation

- The ***North Atlantic Oscillation (NAO)*** is a leading mode of variability of the Northern Hemisphere and beyond.
- It affects ***the atmosphere and oceans*** on several time and space scales.
- Its ***predictive understanding*** could help interannual and decadal-scale climate prediction over and around the North Atlantic basin.
- The ***hierarchical modeling*** approach allows one to give proper weight to the understanding provided by the models *vs*. their realism, respectively.
- Back-and-forth between ***"toy"*** (conceptual) and ***detailed*** ("realistic") ***models***, and between ***models*** and ***data***.

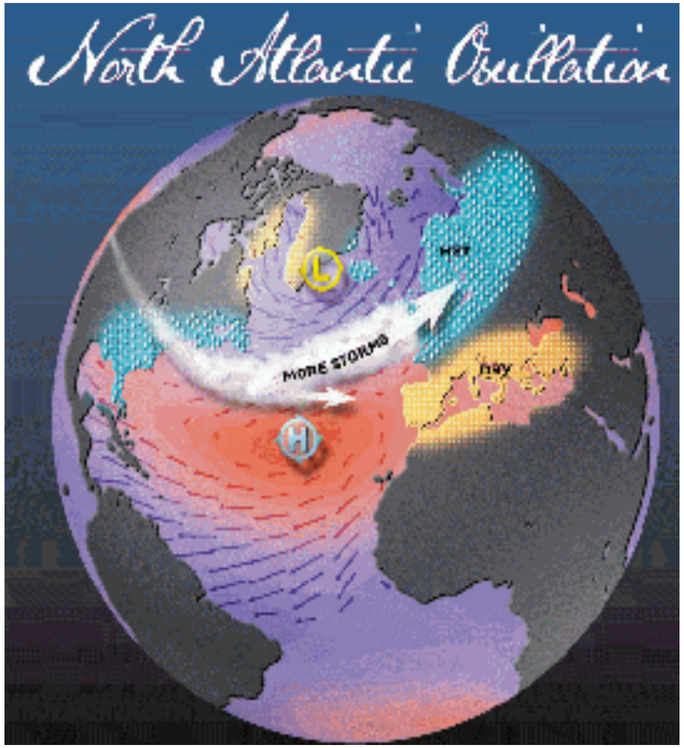
Joint work with *S. Brachet* (SHOM), *F. Codron* (LMD), *H.A. Dijkstra* (Utrecht U.), *Y. Feliks* (IIBR), *S. Jiang (Wall Street), F.-F. Jin* (U. Hawaii), *H. Le Treut* (IPSL), *A.W. Robertson* (Columbia U.), *E. Simonnet* (INLN), *S. Speich* (ENS) & *L. Sushama* (UQAM)

Outline

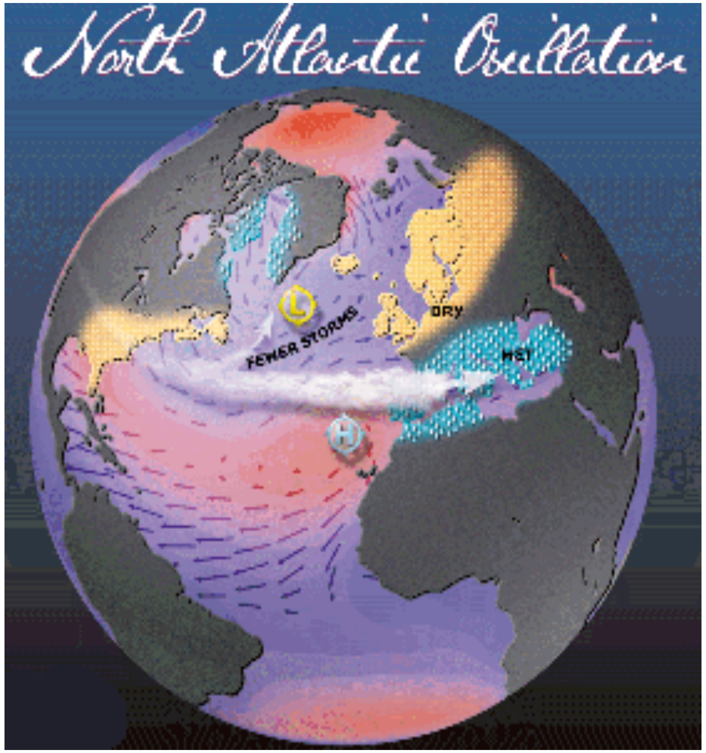
- Introduction: the NAO and the oceans' wind-driven circulation
- The low-frequency variability
  - bifurcations in a toy model
    - => multiple equilibria, periodic and chaotic solutions
  - some intermediate model results
- Atmospheric impacts
  - simple and intermediate models + GCMs
- Some data analysis – atmospheric and oceanic
- Some promising NAO results
- Time-dependent forcing and pullback attractors
- Conclusions
  - The coupled climate system: is it the tail or the dog?
  - Natural climate variability: a source of decadal predictability
- Some references

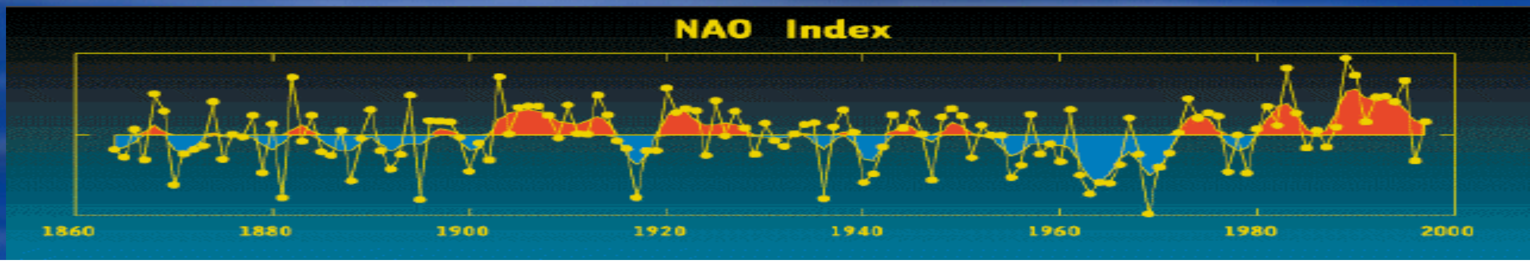
The North Atlantic Oscillation (NAO)

Positive phase

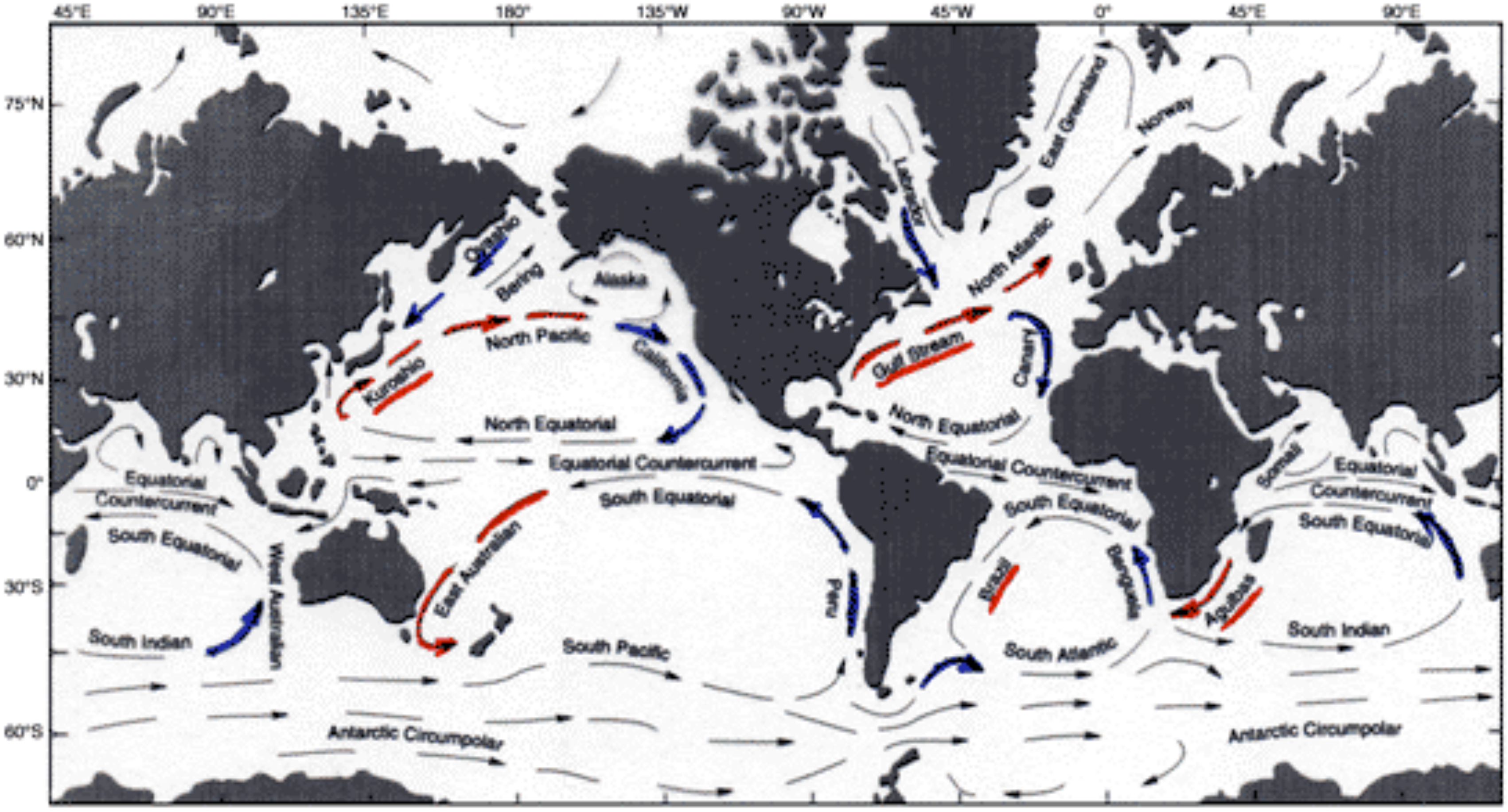


Negative phase





An example of bifurcations and hierarchical modeling: The oceans' wind-driven circulation



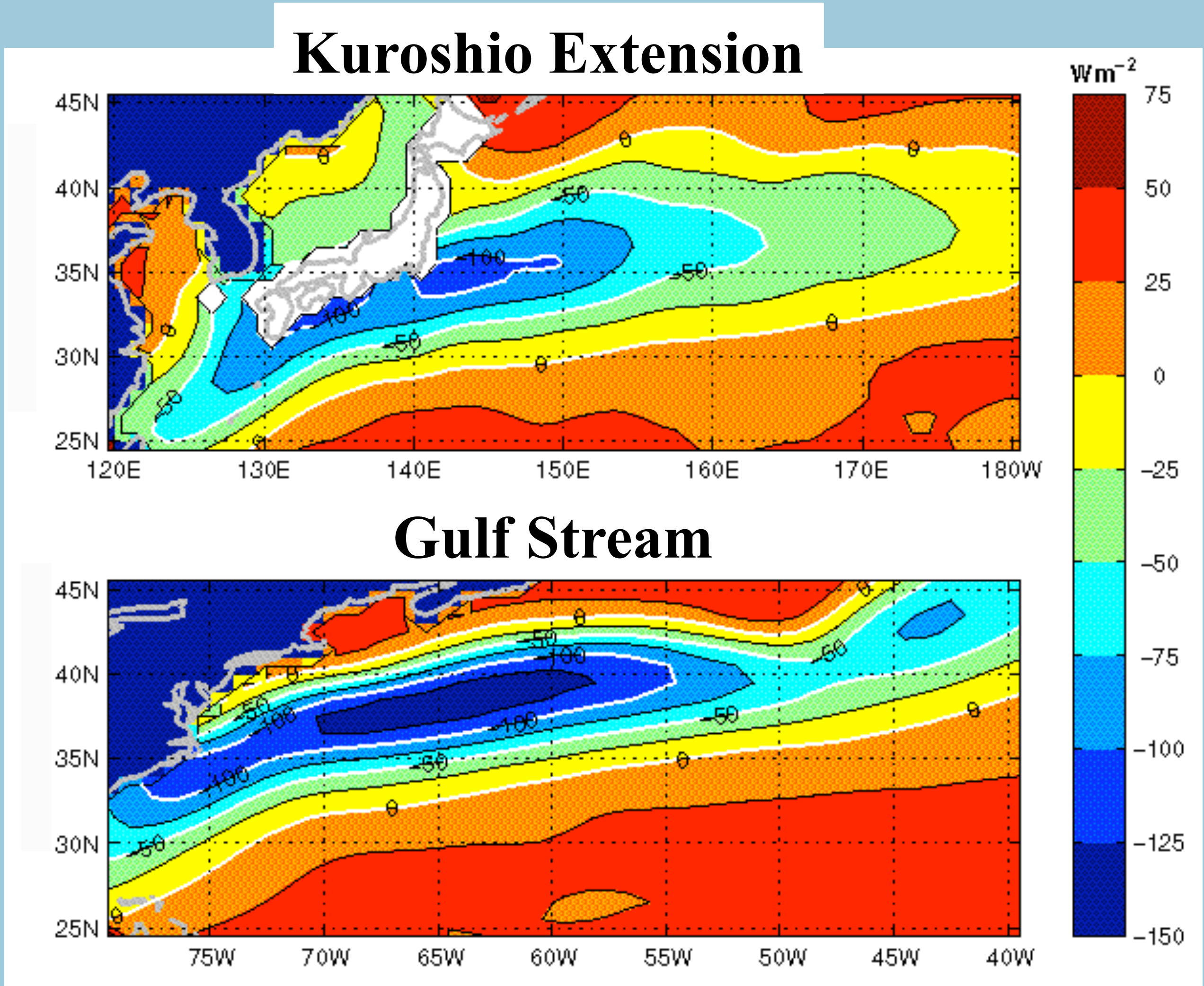
J. Apel (1987), Principles of Ocean Physics

The mean surface currents are (largely) wind-driven

Annual Mean Net Surface Heat Flux

Large heat loss balanced by poleward heat transport (advection)

Latent heat flux is large relative to sensible.



Southhampton Oceanography Centre

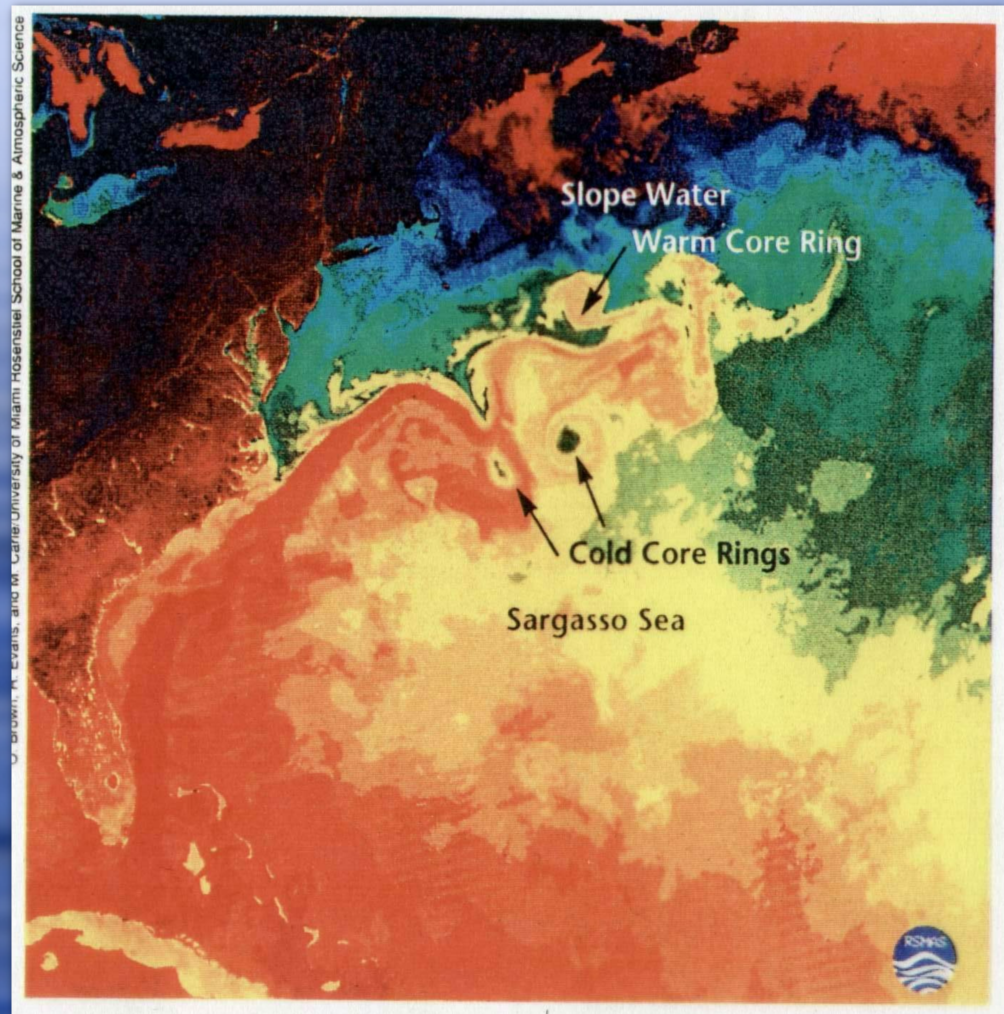
Kelly, Jan 2009

Applied Physics Laboratory
University of Washington

The gyres and the eddies

Many scales of motion, dominated in the mid-latitudes by (i) ***the double-gyre circulation***; and (ii) ***the rings and eddies***.

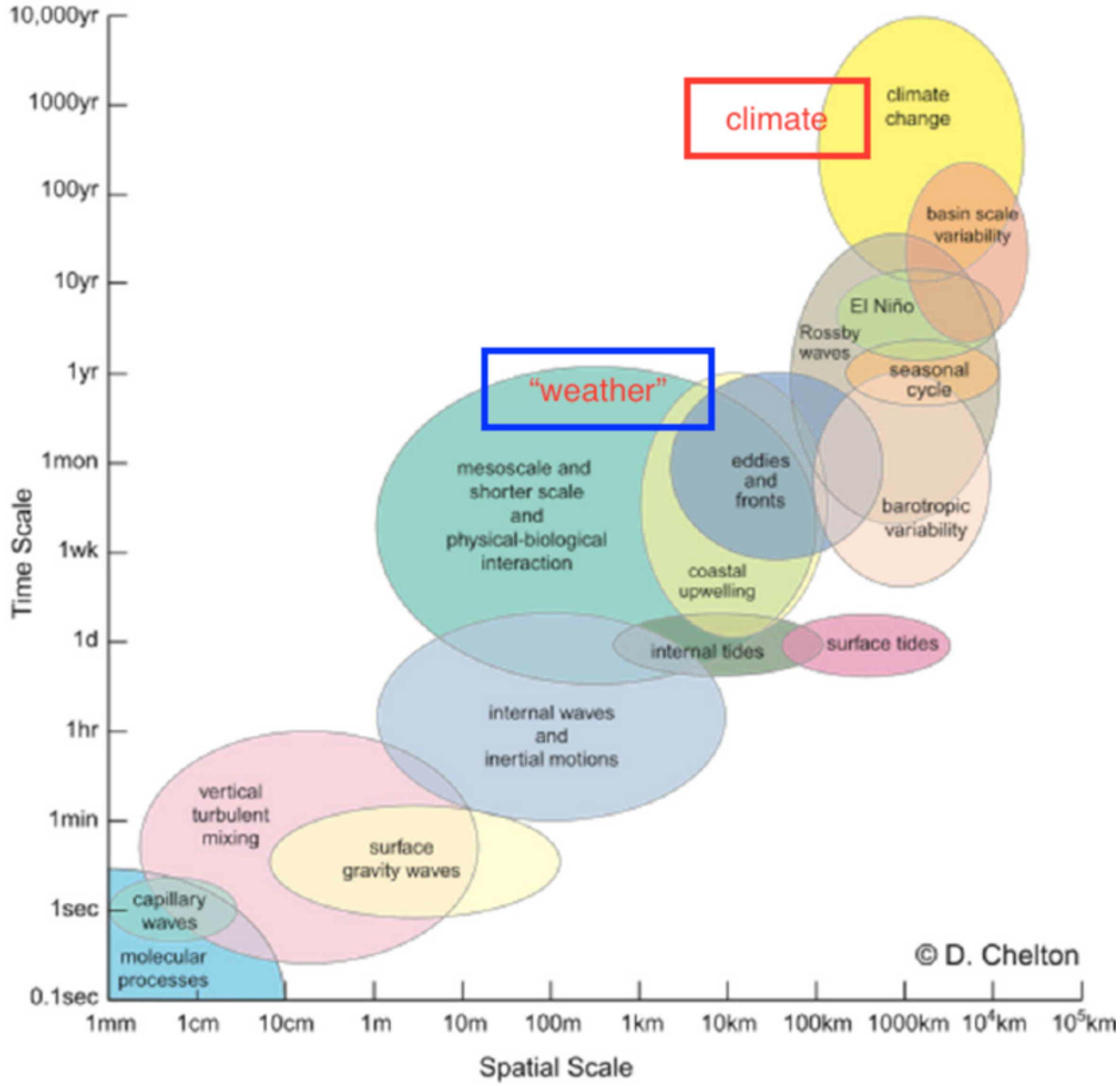
Much of the focus of physical oceanography over the ‘70s to ‘90s has been with the “***meso-scale***”: the meanders, rings & eddies, and the associated two-dimensional and quasi-geostrophic ***turbulence***.



Based on SSTs, from satellite IR data

Space-time organization: oceans

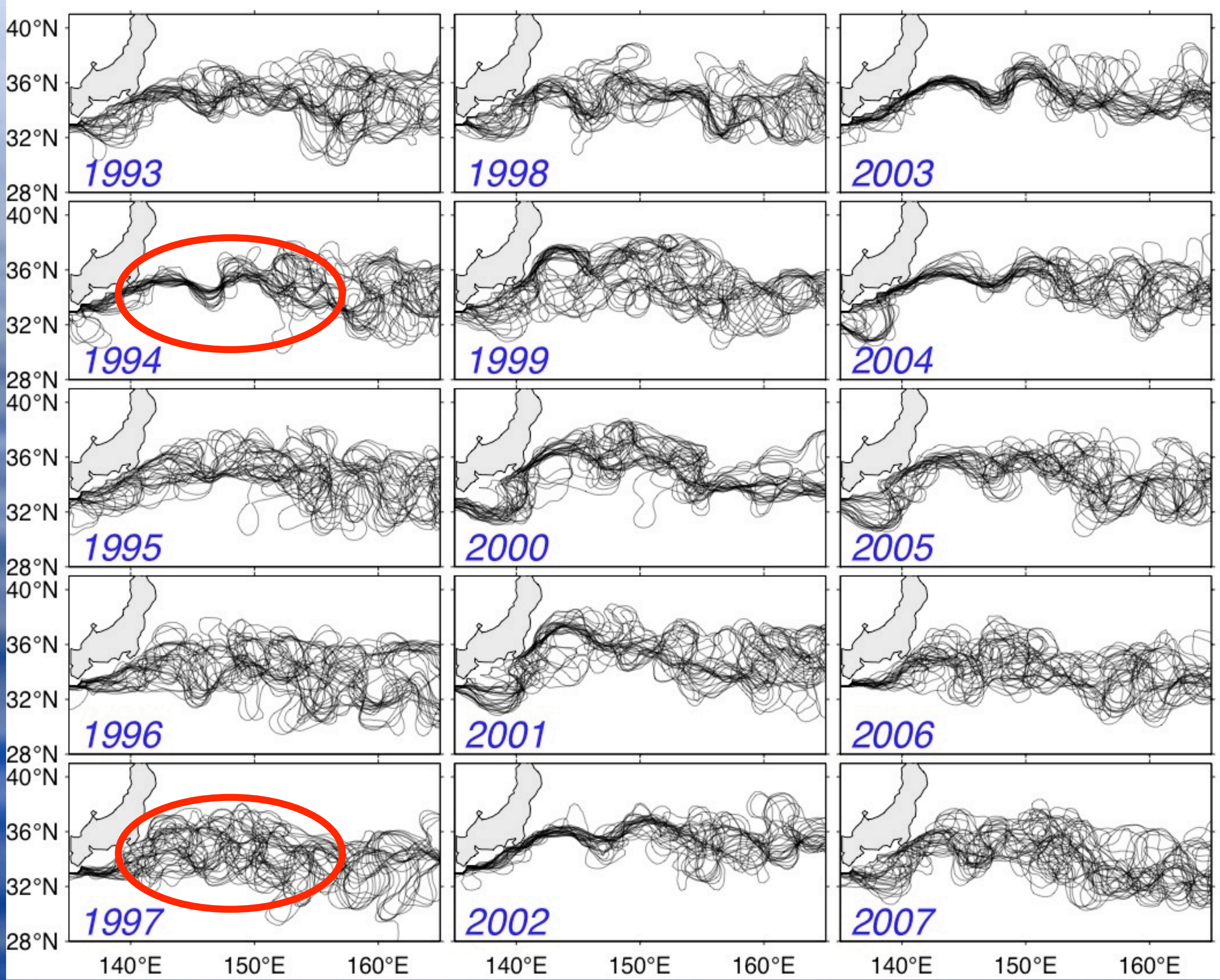
More complex than for the atmosphere, but still, basically,
faster goes with shorter → "weather"
and slower goes with longer → "climate."



Kuroshio Extension (KE) Path Changes

Monthly paths from altimeter:

Stable vs. unstable periods



Qiu & Chen
(*Deep-Sea Res.*, 2009)

"Limited-contour" analysis for atmospheric low-frequency variability

10-day sequences of
subtropical jet paths:
blocked vs. zonal
flow regimes

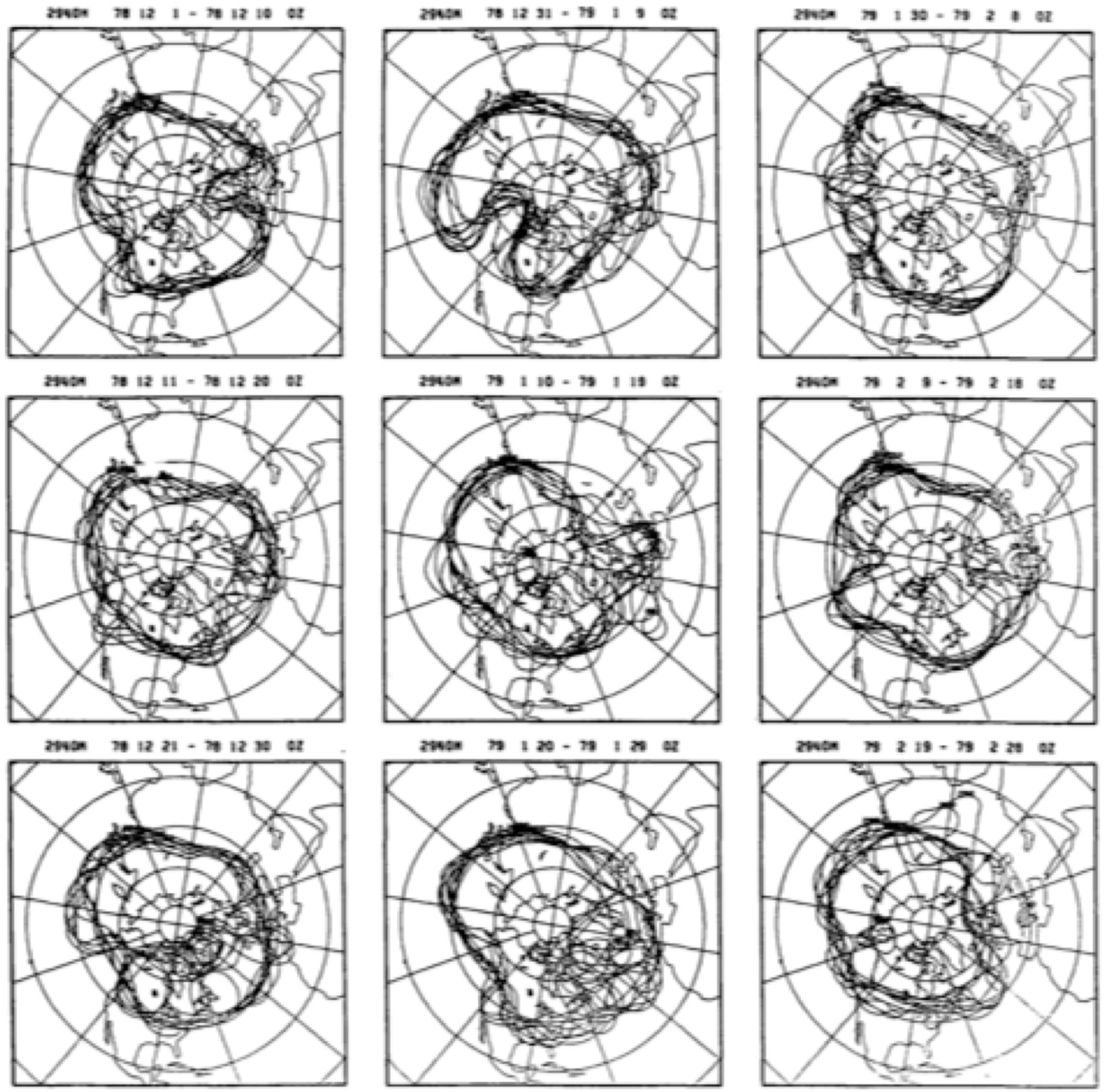
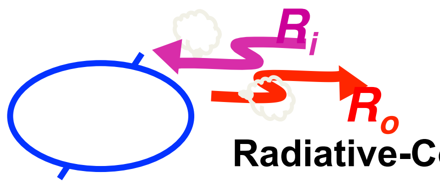


FIG. 1. Limited contour analysis of Northern Hemisphere (NH) flows. Daily contours of a prescribed height (2940 m in this case—roughly corresponding to the jet axis) are superimposed for successive 10-day intervals during NH winter 1978/79. Persistence is illustrated by some of the panels (see text for details).

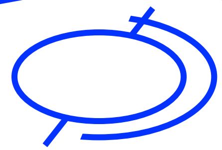
Kimoto & Ghil, *JAS*, 1993a

Climate models (atmospheric & coupled) : A classification

- *Temporal*
  - stationary, (quasi-)equilibrium
  - transient, climate variability

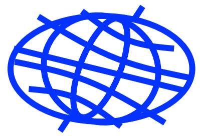


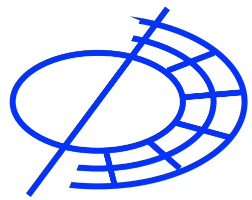
Radiative-Convective Model(*RCM*)



Energy Balance Model (*EBM*)

- *Space*
  - 0-D (dimension 0)
  - 1-D
    - vertical
    - latitudinal
  - 2-D
    - horizontal
    - meridional plane
  - 3-D, GCMs (General Circulation Model)
  - Simple and intermediate 2-D & 3-D models





- *Coupling*
  - Partial
    - unidirectional
    - asynchronous, hybrid
  - Full

➔ ***Hierarchy***: back-and-forth between the simplest and the most elaborate model, and between the models and the observational data

Modeling Hierarchy for the Oceans

Ocean models

- 0-D: box models – chemistry (BGC), paleo
- 1-D: vertical (mixed layer, thermocline)
- 2-D – meridional plane – THC or MOC

  => also 2.5-D: a little longitude dependence

  – horizontal – wind-driven

  => also 2.5-D: reduced-gravity models (*n*.5)
- 3-D: OGCMs – simplified

  – bells & whistles

Coupled 0-A models

- Idealized (0-D & 1-D): intermediate couple models (ICMs)
- Hybrid (HCMs) – diagnostic–statistical atmosphere

  – highly resolved ocean
- Coupled GCMs (3-D): CGCMs

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The double-gyre circulation and its low-frequency variability

An "intermediate" model of the mid-latitude, wind-driven ocean circulation: 20-km resolution, about 15 000 variables

Shallow-water model

$$\frac{\partial U}{\partial t} + \nabla\cdot(\mathbf{u}U) = -g'h\frac{\partial h}{\partial x} + fV + \alpha_A A\nabla^2 U - RU - \alpha_\tau\frac{\tau^x}{\rho}$$

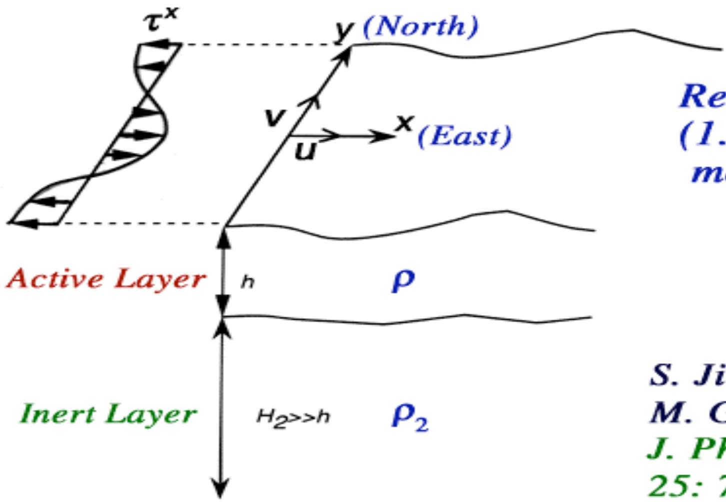
$$\frac{\partial V}{\partial t} + \nabla\cdot(\mathbf{u}V) = -g'h\frac{\partial h}{\partial y} - fU + \alpha_A A\nabla^2 V - RV$$

$$\frac{\partial h}{\partial t} = -\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)$$

where

$$U\hat{e_x} + V\hat{e_y} = h\mathbf{u} = h(u\hat{e_x} + v\hat{e_y})$$

g':	reduced gravity	$(= g(\rho_2 - \rho)/\rho)$
A:	viscosity coefficient	$(= 300\ \mathrm{m^2 s^{-1}})$
R:	Rayleigh coefficient	$(= 1/200\ \mathrm{day^{-1}})$
τ^x:	wind stress $= \tau_o \cos 2\pi/L$ $(\tau_o = 1\ \mathrm{dyn\ cm^{-2}}\ \&\ L = 2000\mathrm{km})$	



Reduced gravity (1.5 - layer) model

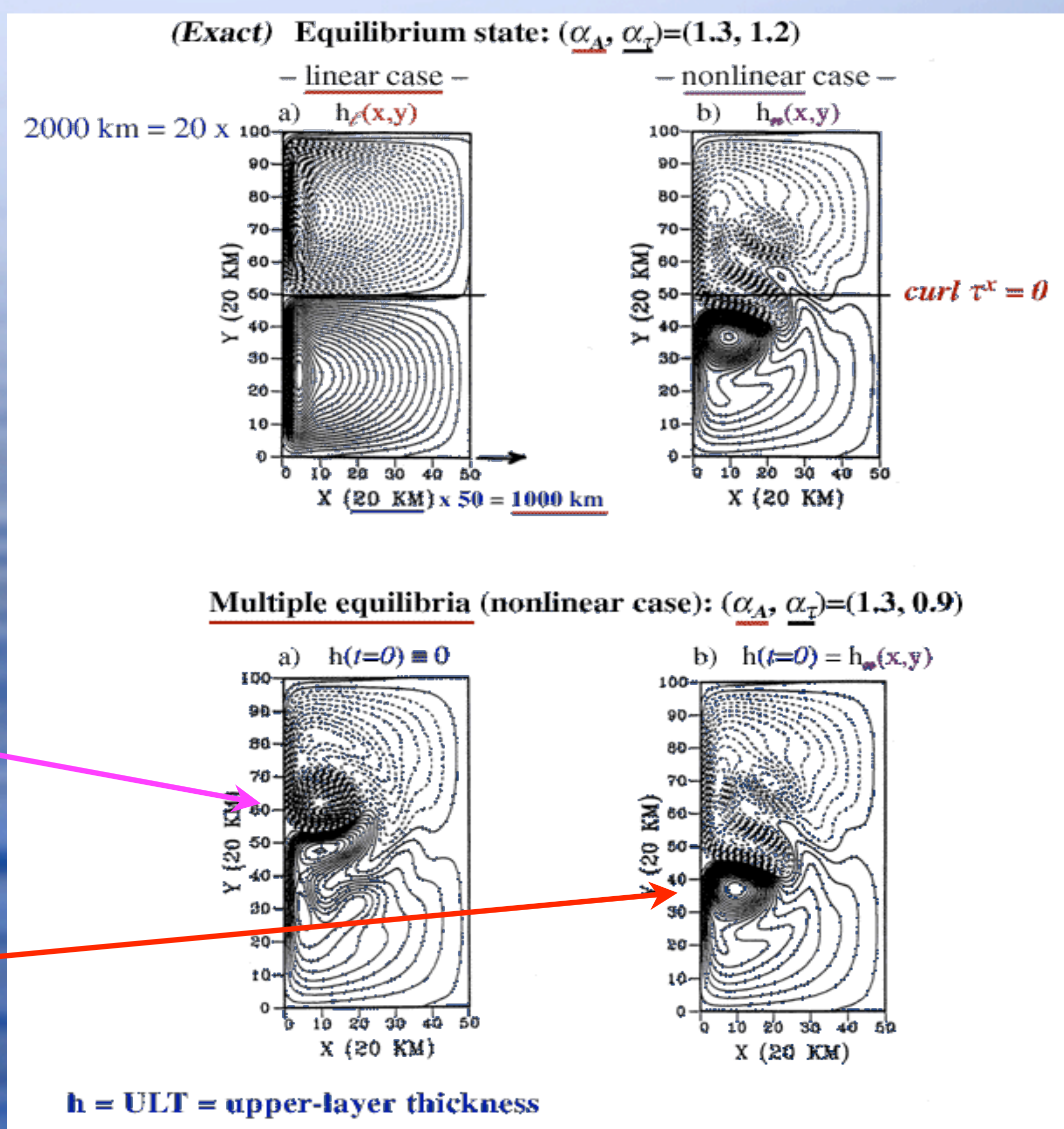
S. Jiang, F.-F. Jin & M. Ghil (1995) J. Phys. Oceanog., 25: 764-786

The JJG model's equilibria

Nonlinear (advection) effects break the (near) symmetry: (perturbed) pitchfork bifurcation?

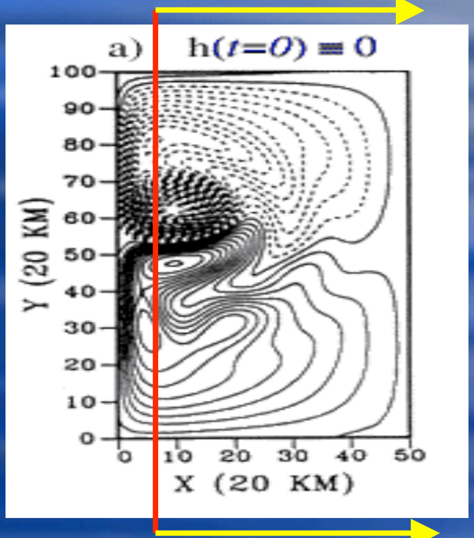
Subpolar gyre dominates

Subtropical gyre dominates



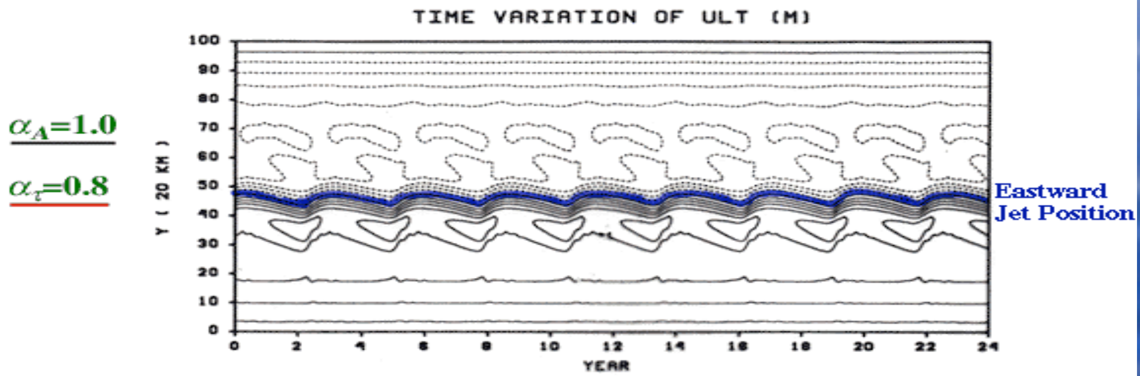
Time-dependent solutions: periodic and chaotic

To capture space-time dependence, meteorologists and oceanographers often use Hovmöller diagrams

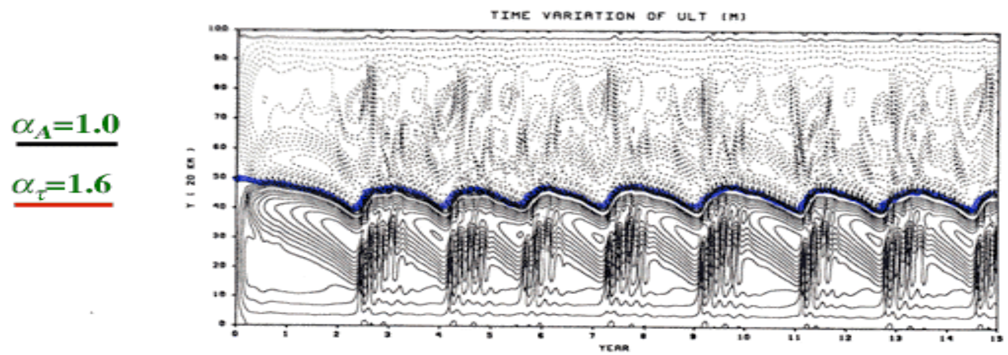


Time-dependent solutions

1. Periodic, w/ interannual period (2.8 years)



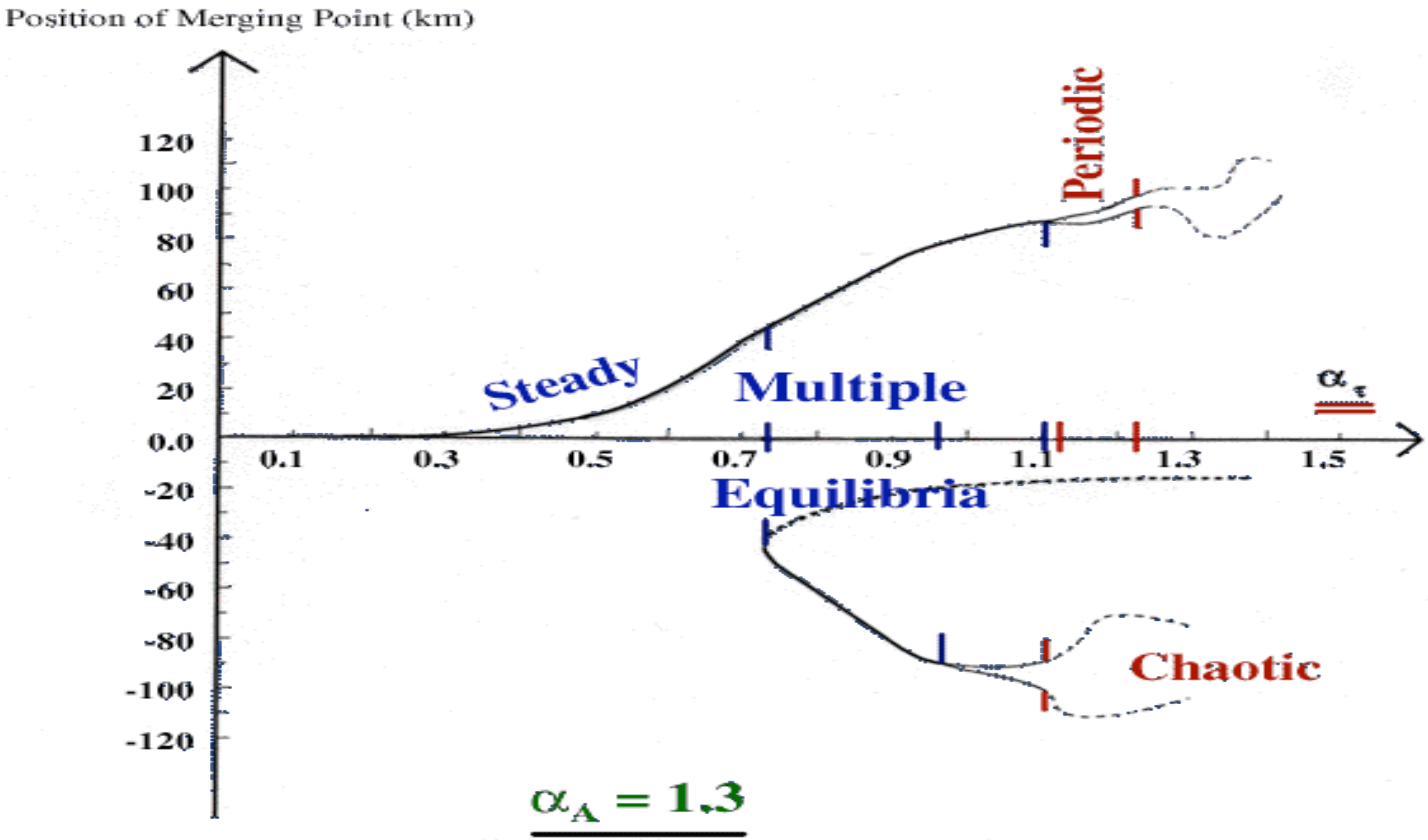
2. Aperiodic (weakly chaotic)



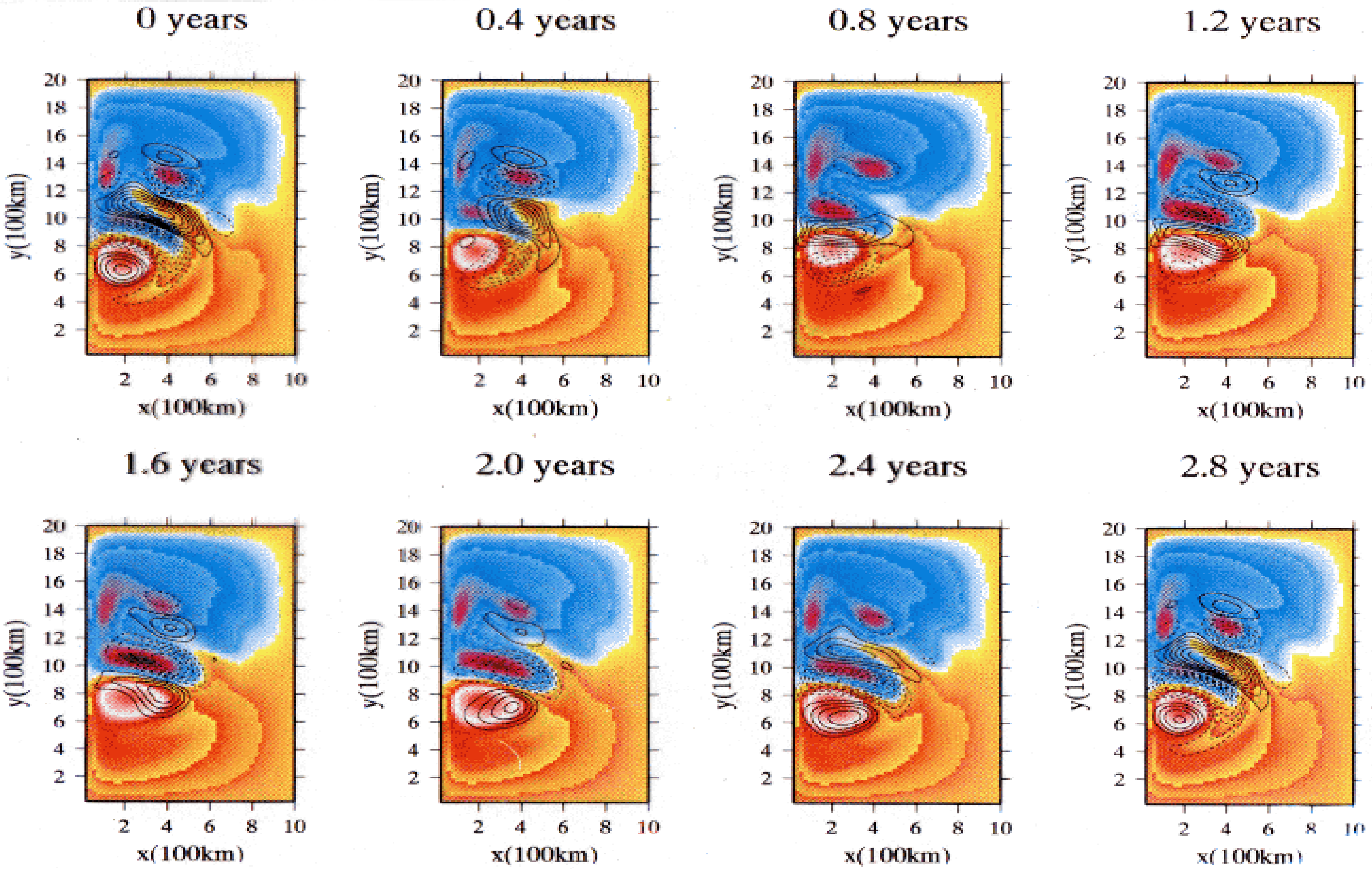
Poor man's continuation method

Bifurcation diagram

Perturbed pitchfork + Hopf + transition to chaos



Interannual variability: relaxation oscillation



Global bifurcations in "intermediate" models

Bifurcation tree in a QG, equivalent-barotropic, high-resolution (10 km) model: pitchfork, mode-merging, Hopf, and homoclinic

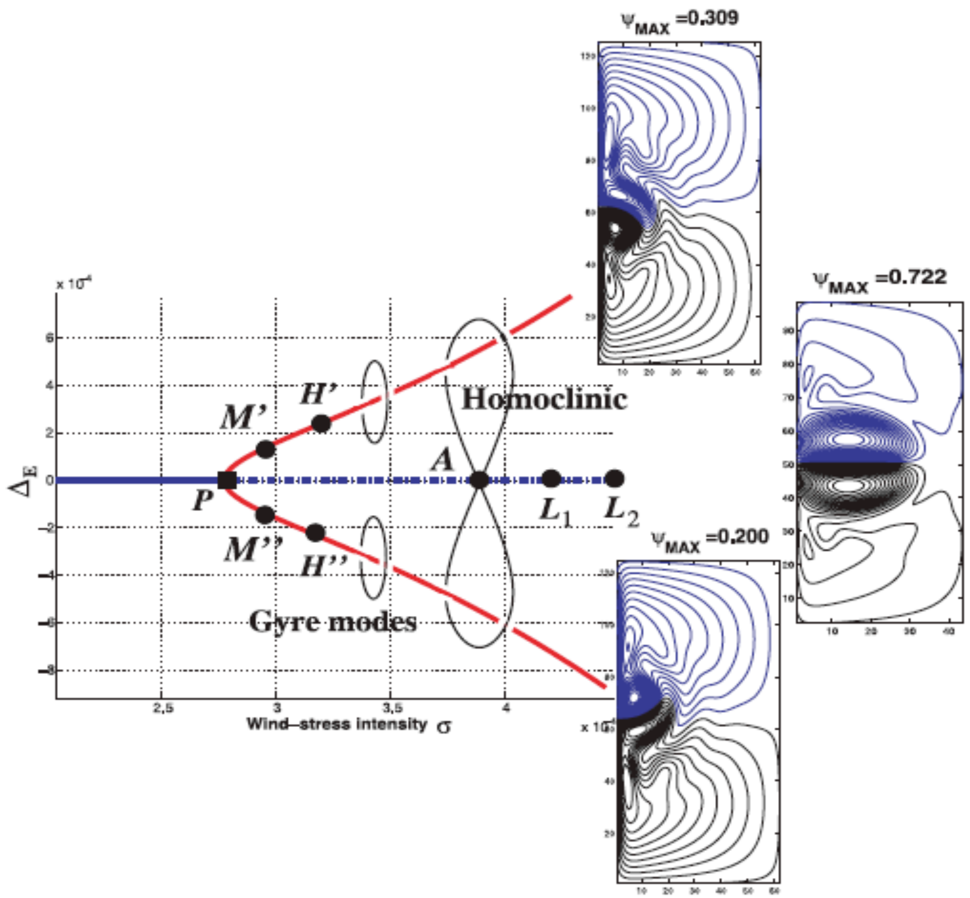


Figure 1. Schematic bifurcation diagram of an equivalent-barotropic QG model, plotted in terms of an asymmetry measure Δ_E (see Section 3a further below) vs. wind-stress intensity. The limit cycles are schematically drawn for illustrative purpose and the streamfunction patterns corresponding to the three steady-state branches—subtropical, antisymmetric, and subpolar (from top to

Homoclinic orbit: numerical and analytical

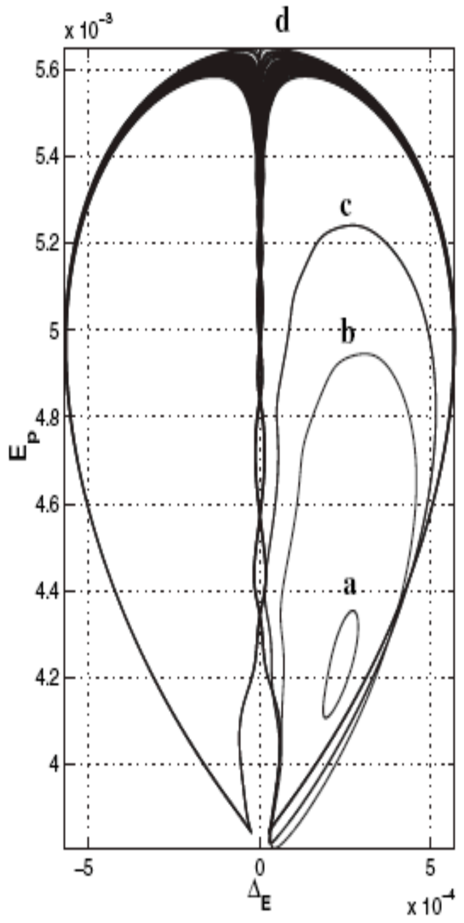


Figure 2. Unfolding of the relaxation oscillations induced by the gyre modes, shown in the plane spanned by the total potential energy of the solution E_p and the difference Δ_E between the subpolar potential energy and the subtropical one (see text for details). The orbits of several limit cycles are

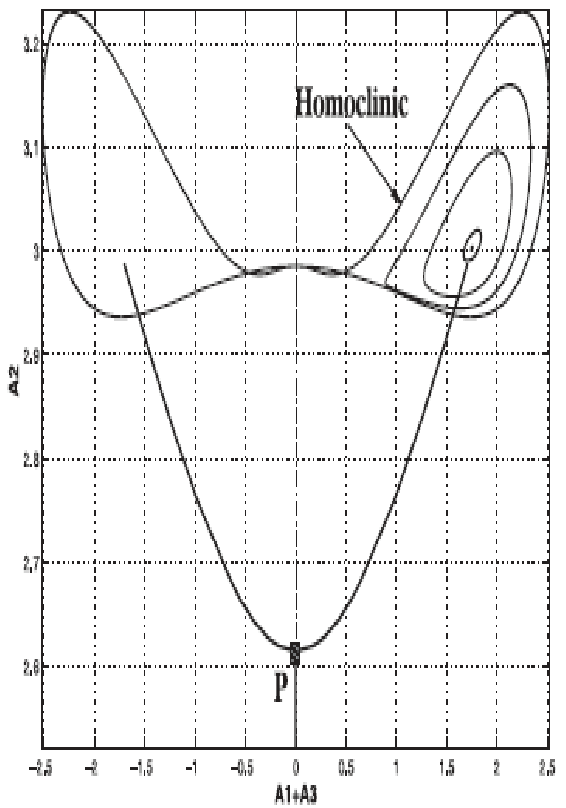


Figure 3. Bifurcation diagram of the highly truncated, four-mode model (5), projected onto the $(A_1 + A_3, A_2)$ plane for $\mu = 1$ and $s = 2$; P stands for pitchfork bifurcation at $\sigma = \sigma_p = 7.61$, while $\sigma = \sigma_{hc} \simeq 10.4299$ at the homoclinic bifurcation. The branches of periodic orbits are replaced by several explicitly computed limit cycles.

The double-gyre circulation: A different rung of the hierarchy

Another “intermediate” model of the double-gyre circulation: slightly different physics, higher resolution – down to 10 km in the horizontal and more layers in the vertical, much larger domain, …

Bo Qiu, U. of Hawaii, pers. commun., 1997

Quasi - geostrophic model

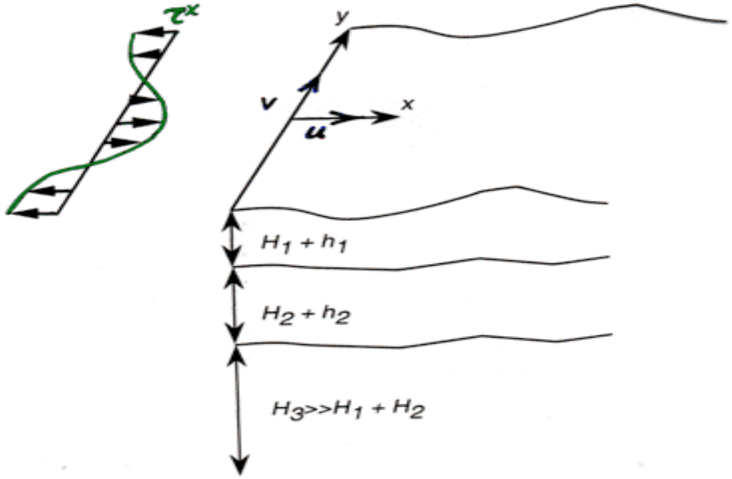
2.5-layer model

$$\frac{\partial}{\partial t}(\nabla^2 h_1 - \lambda_1^2(h_1 - h_2)) + \beta\frac{\partial h_1}{\partial x} = -\frac{g'}{f_0}J[h_1, \nabla^2 h_1 - F_1(h_1 - h_2)] + A_h\nabla^4 h_1 - C\nabla^2(h_1 - h_2) + \frac{f_0}{\rho_0 g' H_1} curl\vec{\tau}$$

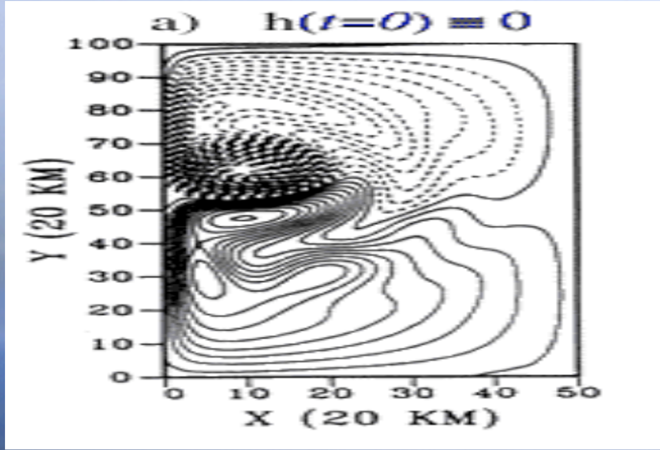
$$\frac{\partial}{\partial t}(\nabla^2 h_2 - \lambda_2^2(h_2 - h_1)) + \beta\frac{\partial h_2}{\partial x} = -\frac{g'}{f_0}J[h_2, \nabla^2 h_2 - F_2(h_2 - h_1)] + A_h\nabla^4 h_2 - C\nabla^2(h_2 - h_1) - R\nabla^2 h_2$$

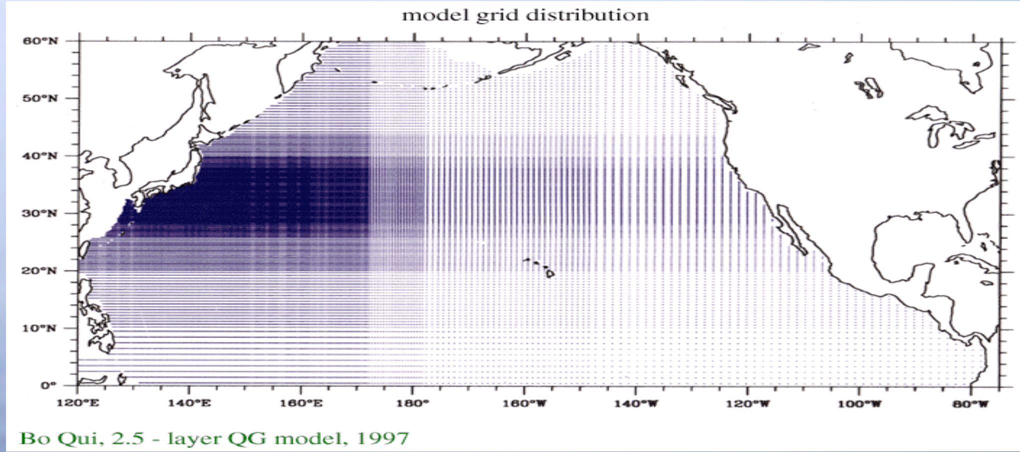
where

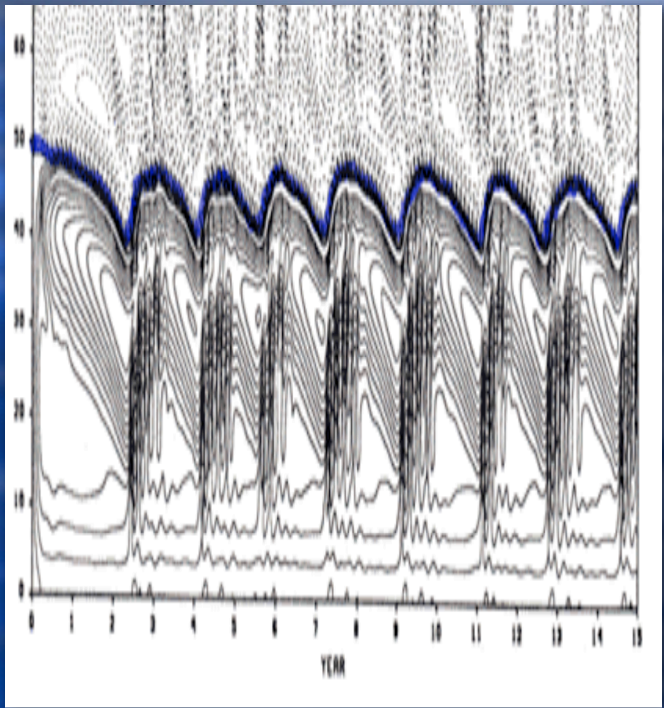
h_1, h_2:	height anomaly for upper and lower layer	(stream functions)
H_1, H_2:	mean height for upper and lower layer	
λ_1, λ_2:	Rossby radius of deformation	$\equiv \sqrt{h'H_1/f_0^2}, \sqrt{h'H_2/f_0^2}$
$\vec{\tau}$:	wind stress	
A_h:	viscosity coefficient	
C, R:	Rayleigh coefficient for interface and lower layer	
f_0, β:	Coriolis and beta parameters	
ρ_0, g':	mean density and reduced gravity	

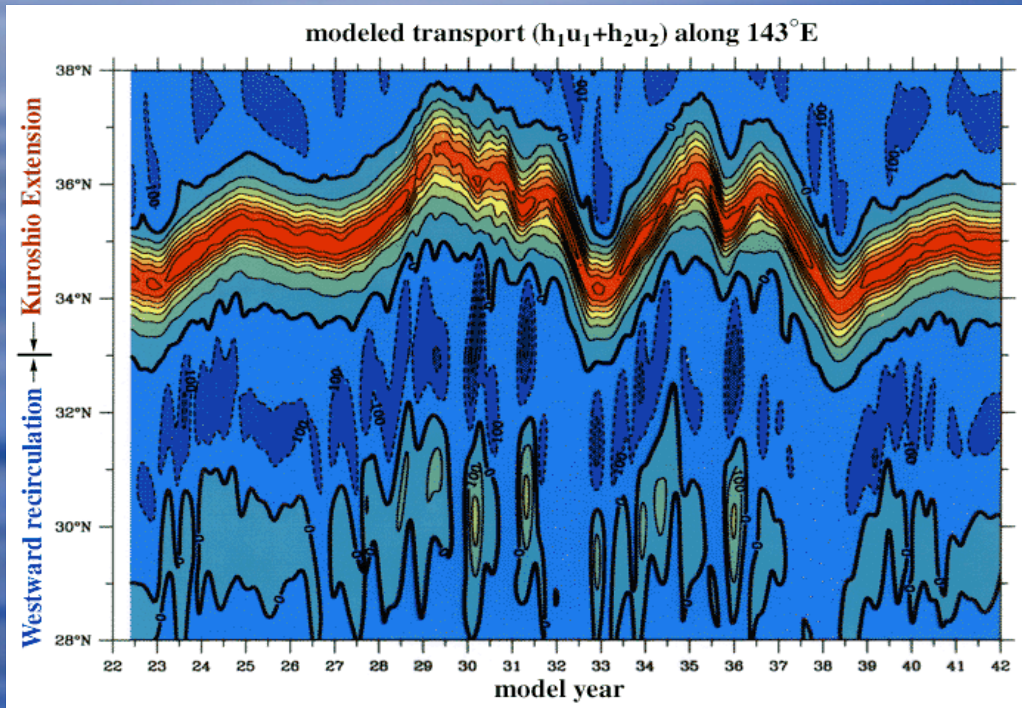


Model-to-model, qualitative comparison









Model-and-observations, quantitative comparison

Spectra of
(a) kinetic energy of 2.5-layer shallow-water model in North-Atlantic–shaped basin; and
(b) Cooperative Ocean-Atmosphere Data Set (COADS) Gulf-Stream axis data

2005] *Simonnet et al.: Quasi-geostrophic double-gyre circulation* 947

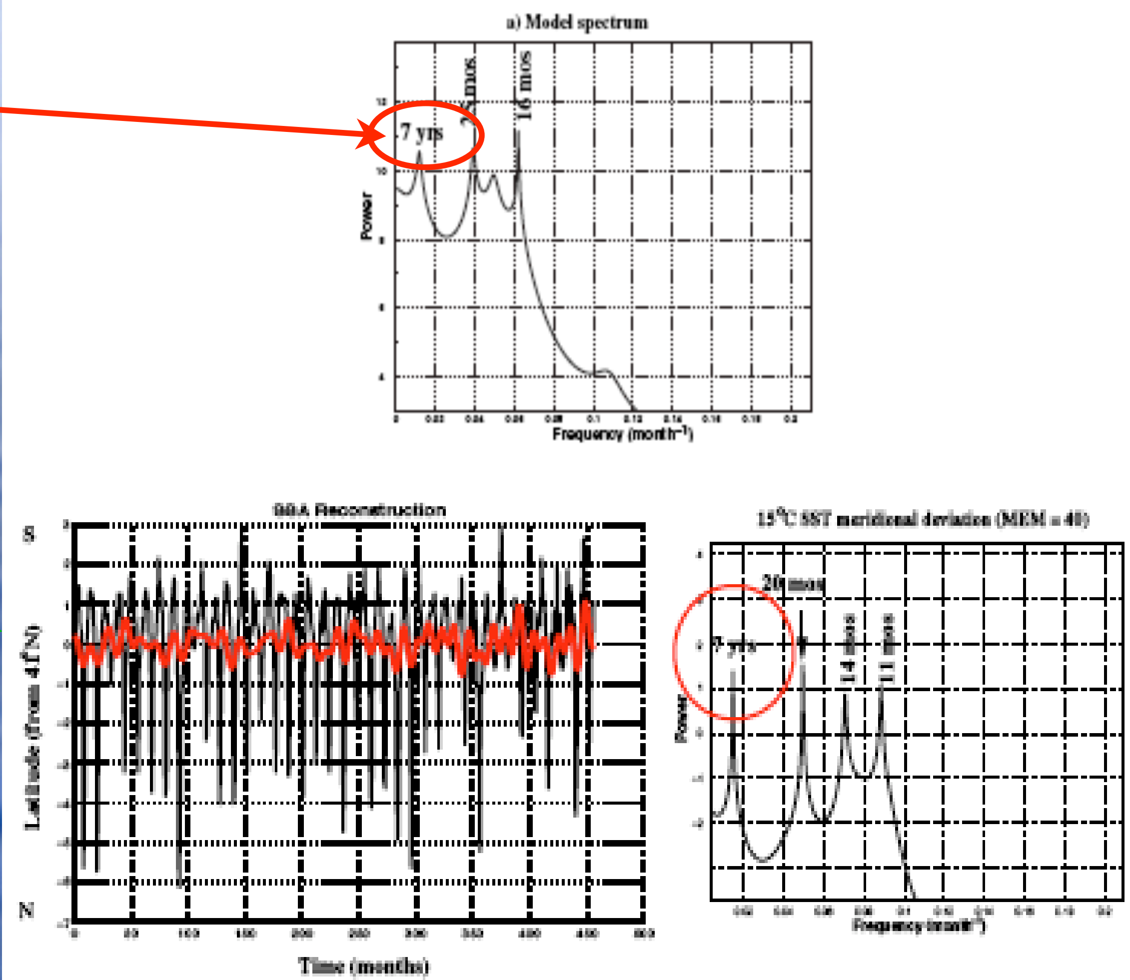


Figure 7. Comparison between low-frequency variability in an idealized double-gyre model and in observations of the Gulf Stream axis. (a) Spectral results for a 2.5-layer SW model for a basin that approximates the North Atlantic in size and shape, using an idealized wind stress. Maximum

More spatio-temporal data

Multi-channel SSA analysis of the UK Met Office monthly mean SSTs for the century-long 1895–1994 interval.

Marked similarity with the 7–8-year "gyre mode" of a full hierarchy of ocean models, on the one hand, and with the North Atlantic Oscillation (NAO), on the other.

Moron, Vautard & Ghil (*Climate Dyn.*, 1988)

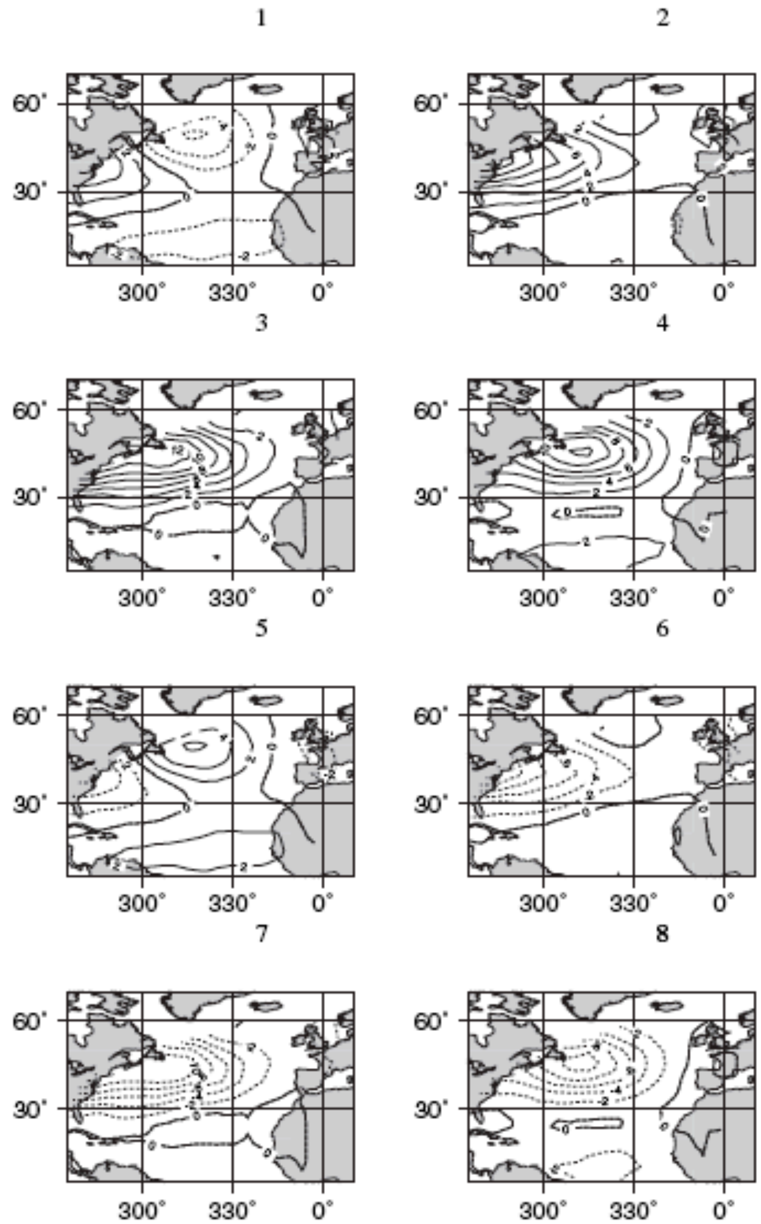


Figure 8. Phase composites of the reconstructed 7–8-year SST oscillation. The MSSA window length is 40 year and the contour interval is 0.02°C.

South Atlantic Observational Data

While there is no fully formed subpolar gyre in the South Atlantic, there is an indication of relaxation oscillations nonetheless:
The separation latitude of the Brazil Current moves slowly away from the Equator and rapidly back toward it.

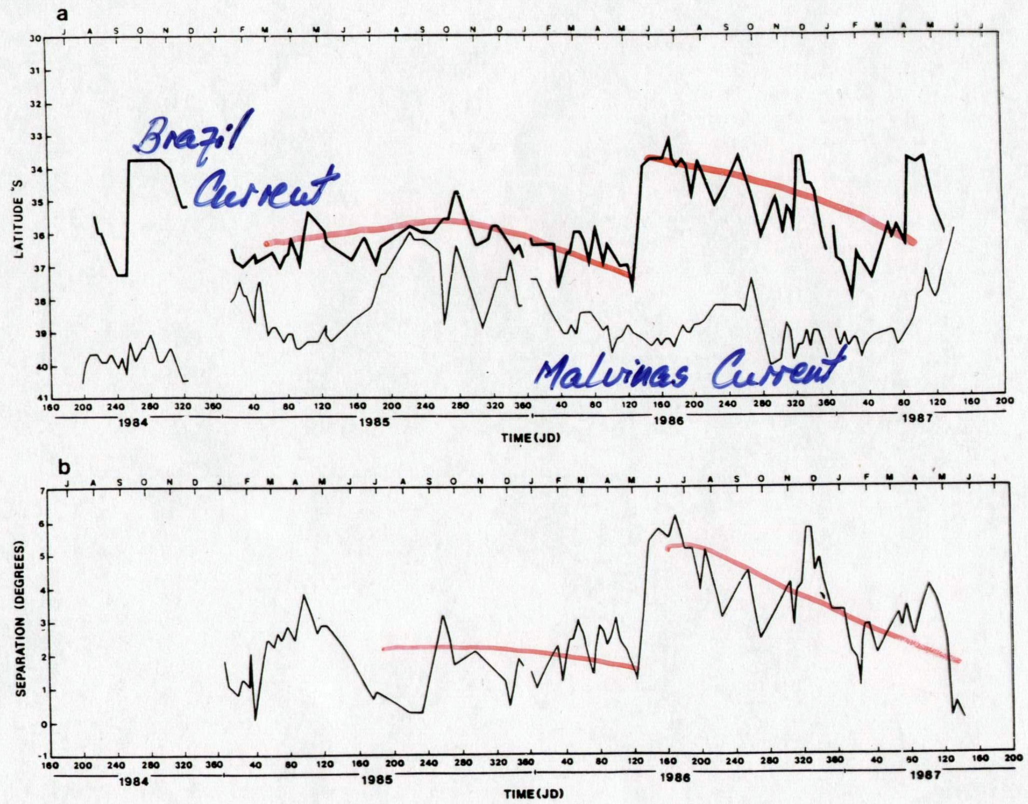


Figure 3.2: Time series of separation latitudes derived from NOAA AVHRR HRPT observations. (a) Brazil Current (thick curve) and Malvinas (thin curve) separation from the 1000 m isobath; (b) the distance between the two separation locations in time. Separations are given in degrees of latitude. Note the abrupt transition of the Brazil Current during the middle of 1986, usually associated with the system's nonlinearity (from Olson et al. 1988).

Olson et al.
(*Deep Sea Res.*, 1988)

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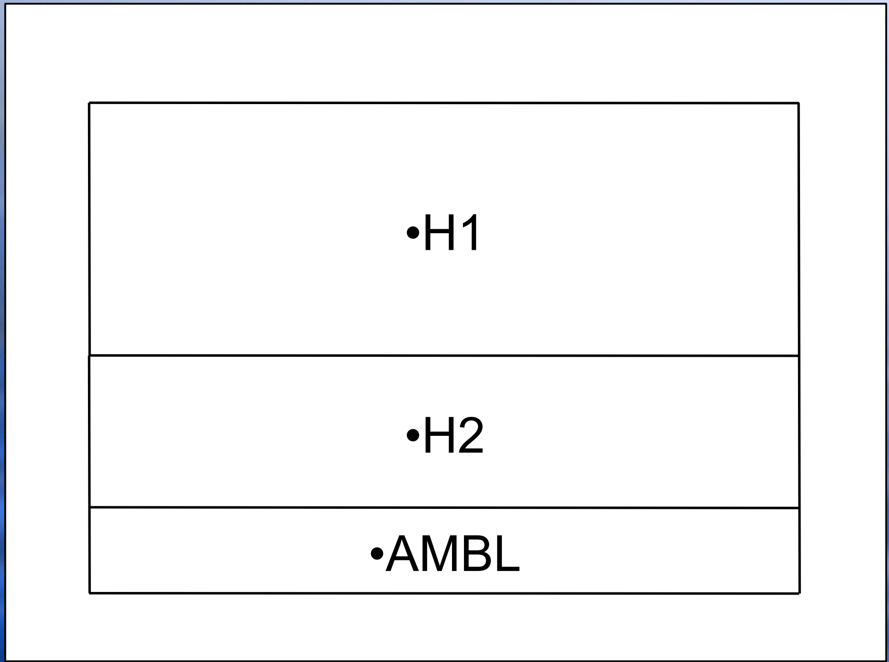
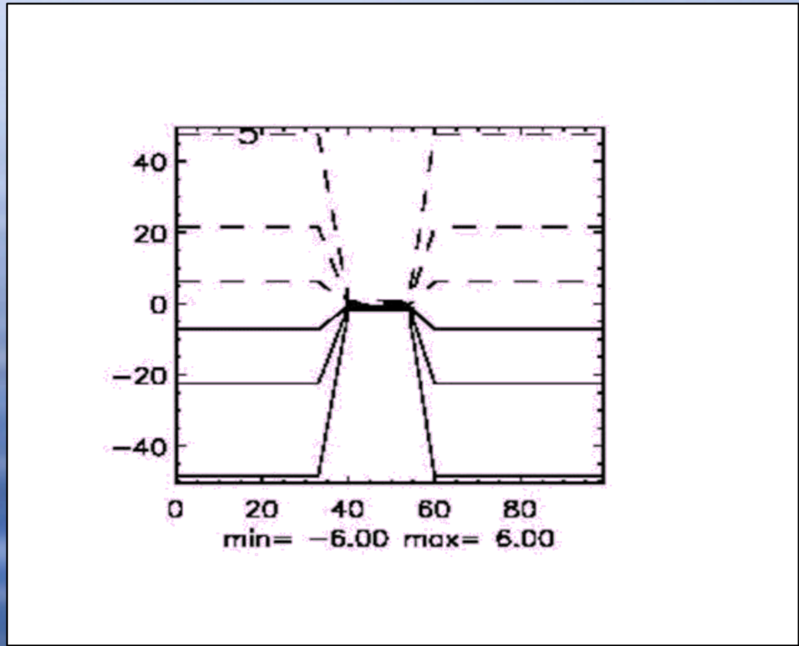
Atmospheric impact of mid-latitude SST anomalies: A highly contentious issue





- A quasi-geostrophic (QG) atmospheric model in a periodic β-channel, first barotropic (Feliks *et al.*, *JAS*, 2004; FGS'04), then baroclinic (FGS'07).
- Marine atmospheric boundary layer (ABL), analytical solution.
- Forcing by idealized oceanic SST front.

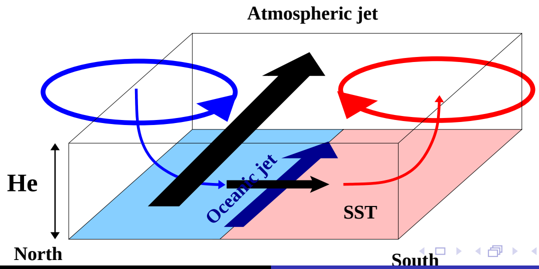
Vertical velocity at the top of the marine ABL

- The nondimensional $w(H_e)$ is given by

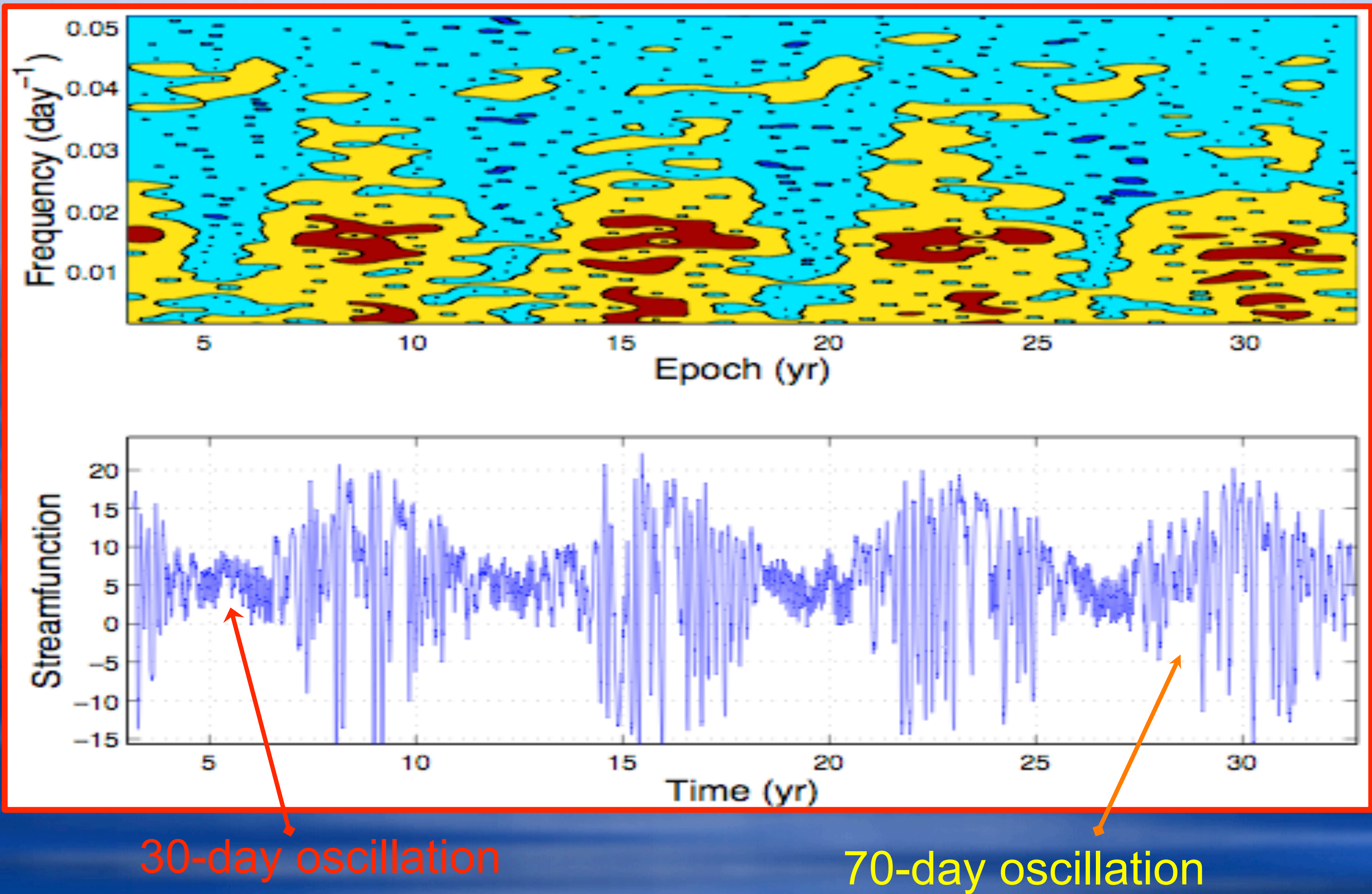
$$w(H_e) = \left[\gamma\zeta_g - \alpha\nabla^2 T\right],$$

with $\gamma = c_1(f_0 L/U)(H_e/H_a)$ and $\alpha = c_2(g/T_0U^2)(H_e^2/H_a)$, where H_a is the layer depth of the free atmosphere ($\sim$ 10 km), and ζ_g the atmospheric geostrophic vorticity.

- Two components: one **mechanical**, due to the geostrophic flow ζ_g above the marine ABL and one **thermal**, induced by the SST front.



Evolutive spectral analysis

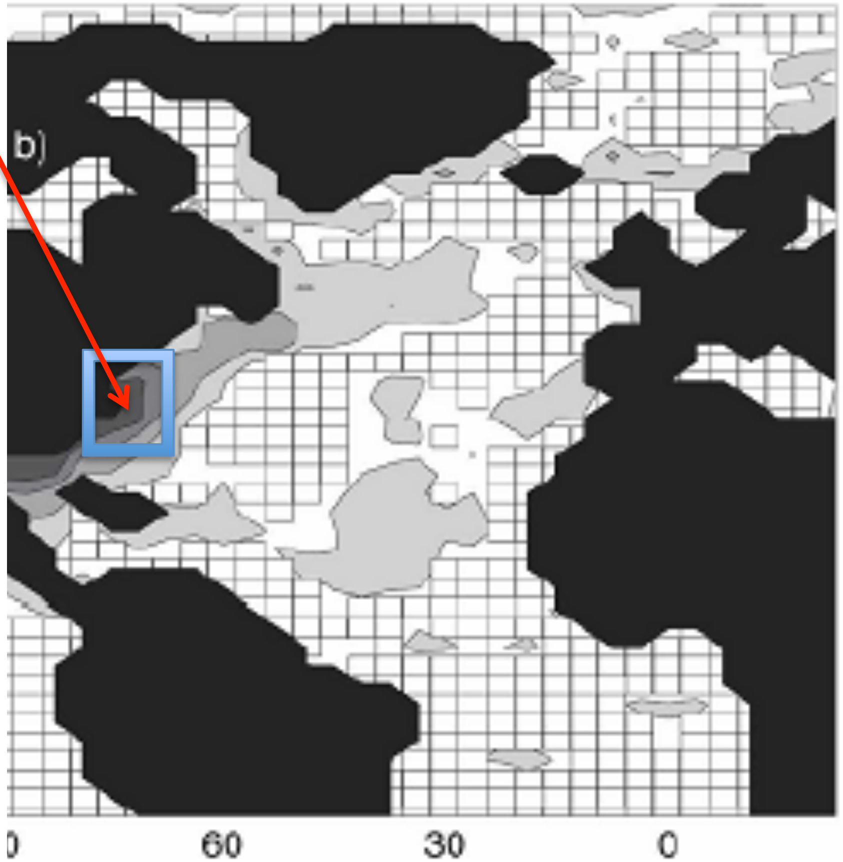


SST effects on NAO, via Granger causality

Q: Where does SST add information to knowledge of the NAO?

A: Just where you'd expect it!

Daily data from
50-yr simulation of
IPCC-class coupled GCM,
HadCM3

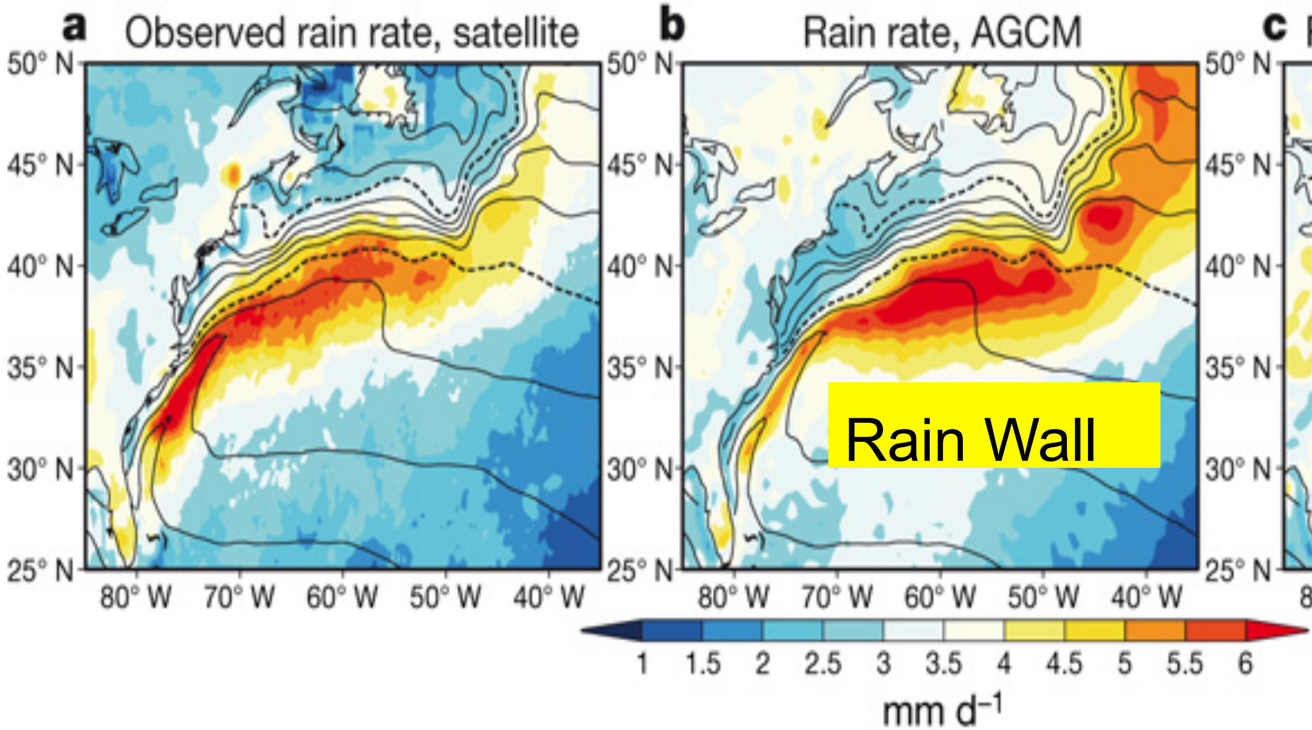


Mosedale, Stephenson, Collins & Mills
(*J. Climate*, 2006)

Precipitation effects:

sat. obs. (TRMM 3B43), ECMWF reanalyses,

and AFES (AGCM for the Earth Simulator, T239, 48 levels)



Minobe *et al*. (*Nature*, 2008):

smoothing the SST field suppresses the rain wall

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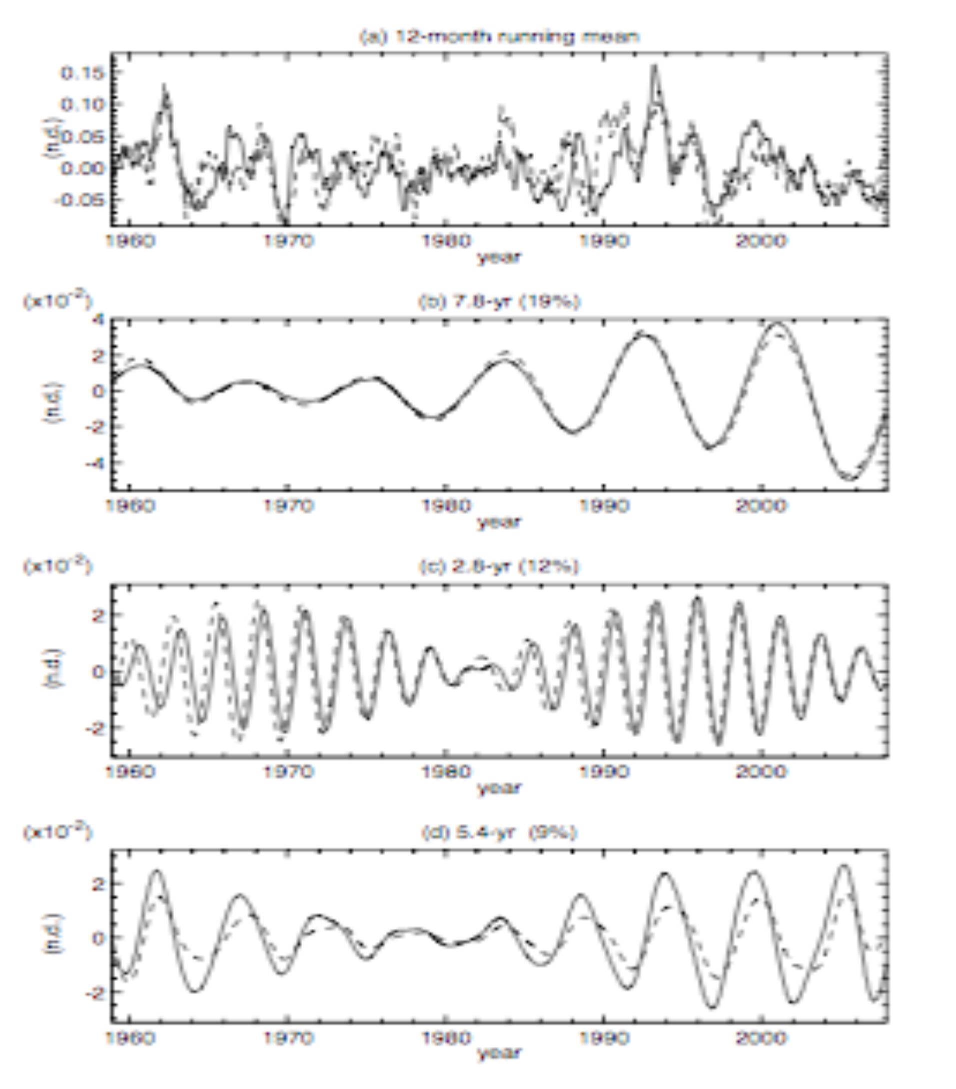
The 7–8-yr mode in atmospheric data
Likewise a contentious issue

Simulate atmospheric response to SODA data over the Gulf Stream region

- Use SST (–5 m) data from the SODA reanalysis (50 years)
- Use the FGS'07 QG model in periodic β-channel
  - – baroclinic + marine ABL
- Figure shows NAO index:
  - – simulated (solid)
  - – observed (dashed)



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Time-dependent forcing, I

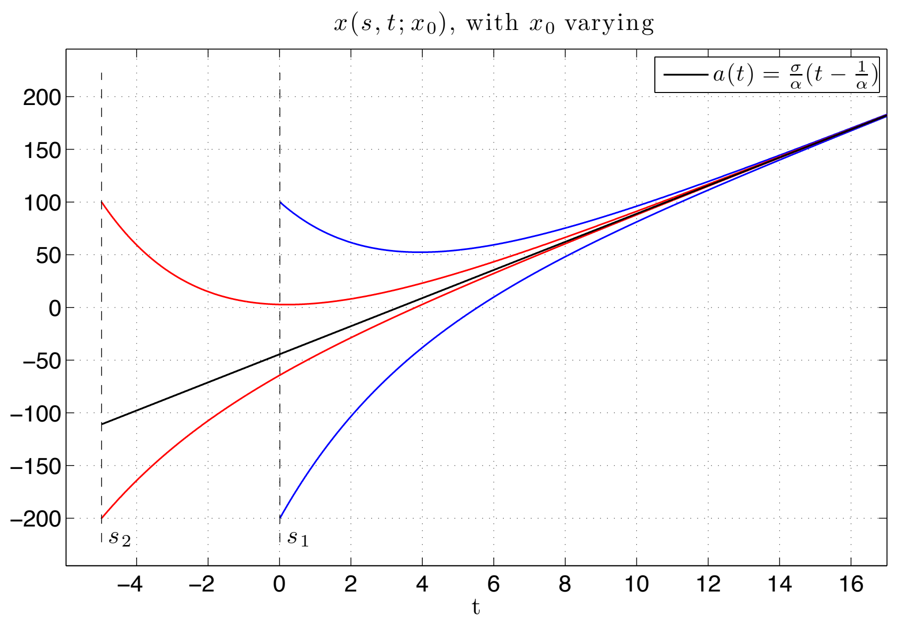
- Much of the theoretical work on the intrinsic variability of the wind-driven ocean circulation has been done with time-independent wind stress.
- To address truly coupled ocean–atmosphere behavior and climate change an important step is to examine time-dependent wind stress.
- The proper framework for doing so is the theory of non-autonomous and random dynamical systems (NDS and RDS).
- We do so here with a "toy" model given by the low-order truncation of the QG, equivalent-barotropic potential vorticity equation (PVE).
- The forcing is deterministic, aperiodic, and dominated by multi-decadal variability.

The pullback attractor of a linear, scalar ODE,

$$\dot{x} = -\alpha x + \sigma t, \quad \alpha > 0, \ \sigma > 0,$$

is given by

$$a(t) = \frac{\sigma}{\alpha}\left(t - \frac{1}{\alpha}\right).$$



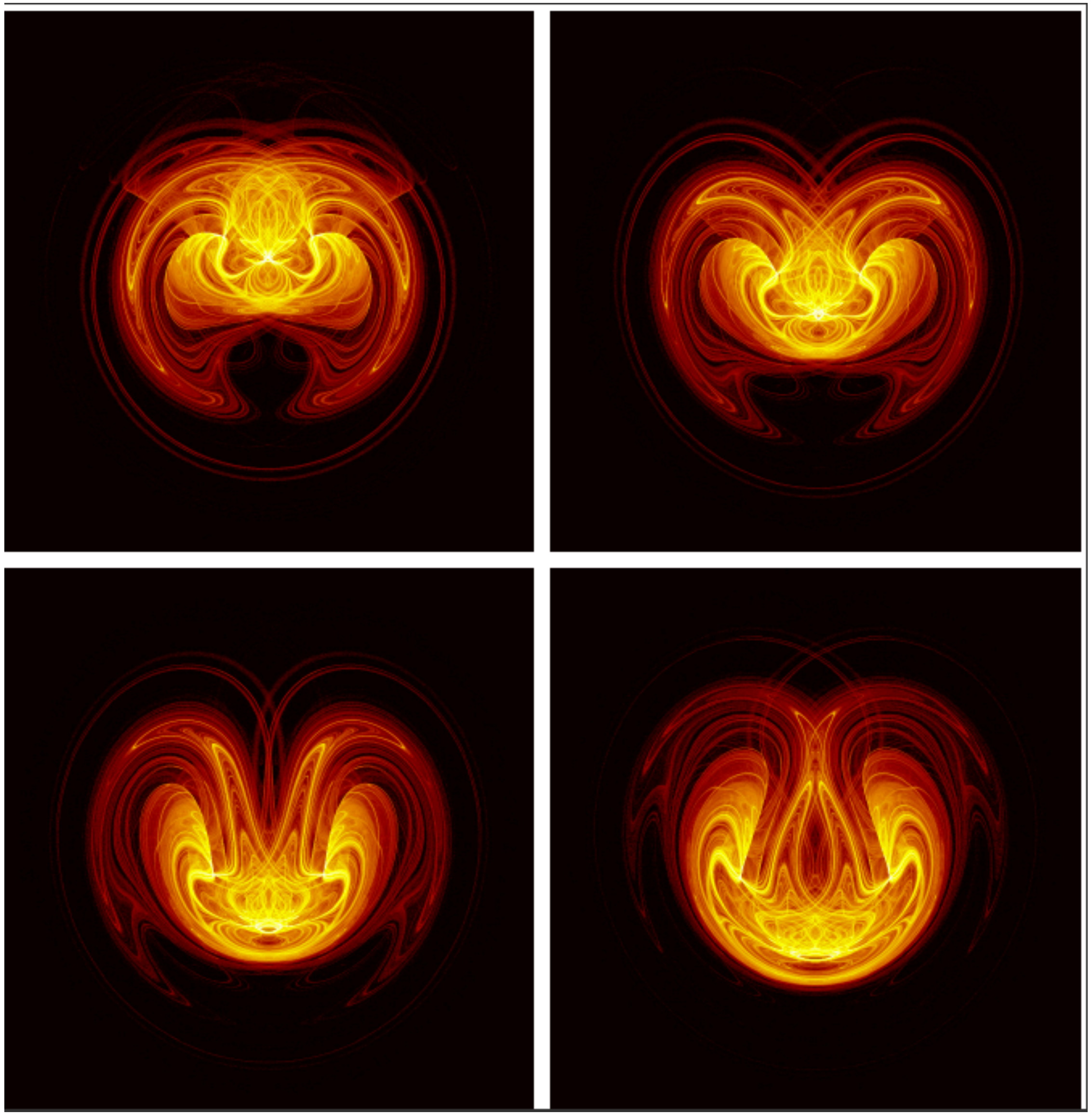
Random Attractor

Physically open system, modeled mathematically as non-autonomous system: allows for deterministic (anthropogenic) as well as random (natural) forcing.

The attractor is "pullback" and evolves in time ~ "imaginary" or "complex" number.

Climate sensitivity ~ change in the statistical properties (first and higher-order moments) of the attractor as one or more parameters (λ, μ, …) change.

Ghil (*Encyclopedia of Atmospheric Sciences*, 2nd ed., 2012)



Atmospheric & Oceanic Sciences
Departmental Seminar Series

McGill U., 12 Jan. 2017

The Wind-Driven Ocean Circulation: Bifurcations, Simulations and Observations

Michael Ghil

Ecole Normale Supérieure, Paris, and
University of California, Los Angeles

Joint work with many people over the years; most recently **M.D. Chekroun** (UCLA) & **S. Pierini** (U. Napoli-Parthenope) + **L. De Cruz, J. Demaeyer** & **S. Vannitsem** (RMI, Brussels)

Please visit these sites for more info.

http://www.atmos.ucla.edu/tcd/

http://www.environnement.ens.fr/

Time-dependent forcing, II

The highly idealized, toy model of the QG, equivalent-barotropic PVE is given by the following system of four quadratically nonlinear ODEs:

$$\dot{\psi}_1 + L_{11}\psi_1 + L_{13}\psi_3 + B_1(\Psi,\Psi) = W_1(t),$$
$$\dot{\psi}_2 + L_{22}\psi_2 + L_{24}\psi_4 + B_2(\Psi,\Psi) = W_2(t),$$
$$\dot{\psi}_3 + L_{33}\psi_3 + L_{31}\psi_1 + B_3(\Psi,\Psi) = W_3(t),$$
$$\dot{\psi}_4 + L_{44}\psi_4 + L_{42}\psi_2 + B_4(\Psi,\Psi) = W_4(t);$$

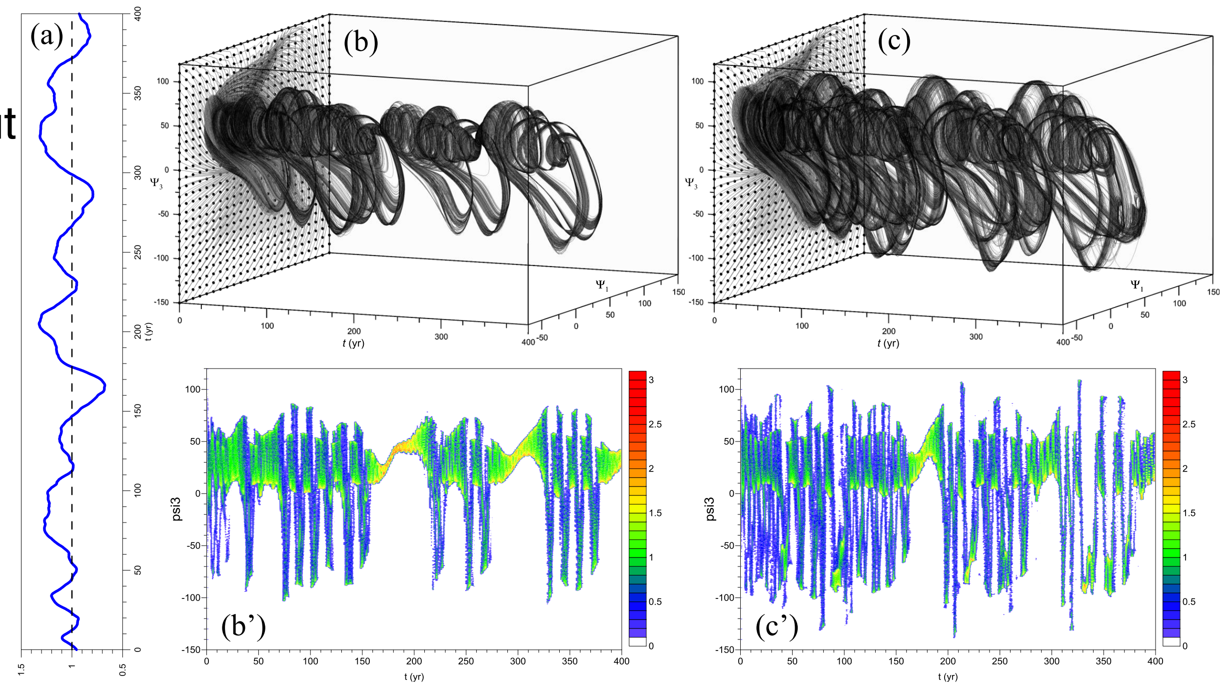
where Ψ denotes the vector $(\psi_1, \psi_2, \psi_3, \psi_4)$ and the bilinear terms B_i are given by

$$B_1(\Psi,\Psi) = 2J_{112}\psi_1\psi_2 + 2J_{114}\psi_1\psi_4 + 2J_{134}\psi_3\psi_4,$$
$$B_2(\Psi,\Psi) = J_{211}\psi_1^2 + J_{222}\psi_2^2 + J_{233}\psi_3^2 + 2J_{213}\psi_1\psi_3 + 2J_{224}\psi_2\psi_4,$$
$$B_3(\Psi,\Psi) = 2J_{314}\psi_1\psi_4 + 2J_{323}\psi_2\psi_3 + 2J_{334}\psi_3\psi_4,$$
$$B_4(\Psi,\Psi) = J_{411}\psi_1^2 + J_{422}\psi_2^2 + J_{433}\psi_3^2 + J_{444}\psi_4^2 + 2J_{413}\psi_1\psi_3 + 2J_{424}\psi_2\psi_4.$$

Time-dependent forcing, III

- The quadratic terms are conservative and the linear terms are weakly dissipative, while the system is unstable for reasonable parameter values.
- For autonomous systems, we know that these properties can lead to chaotic solutions that live on a strange attractor.
- Here they lead to the existence of a pullback attractor (PBA).

Time-dependent forcing at far left + ensemble of 644 orbits starting from the same subset Γ but with different parameter values: left panels, close to periodic; right panels, fully chaotic; lower panels, time series of ψ_3. Time interval is 400 years.

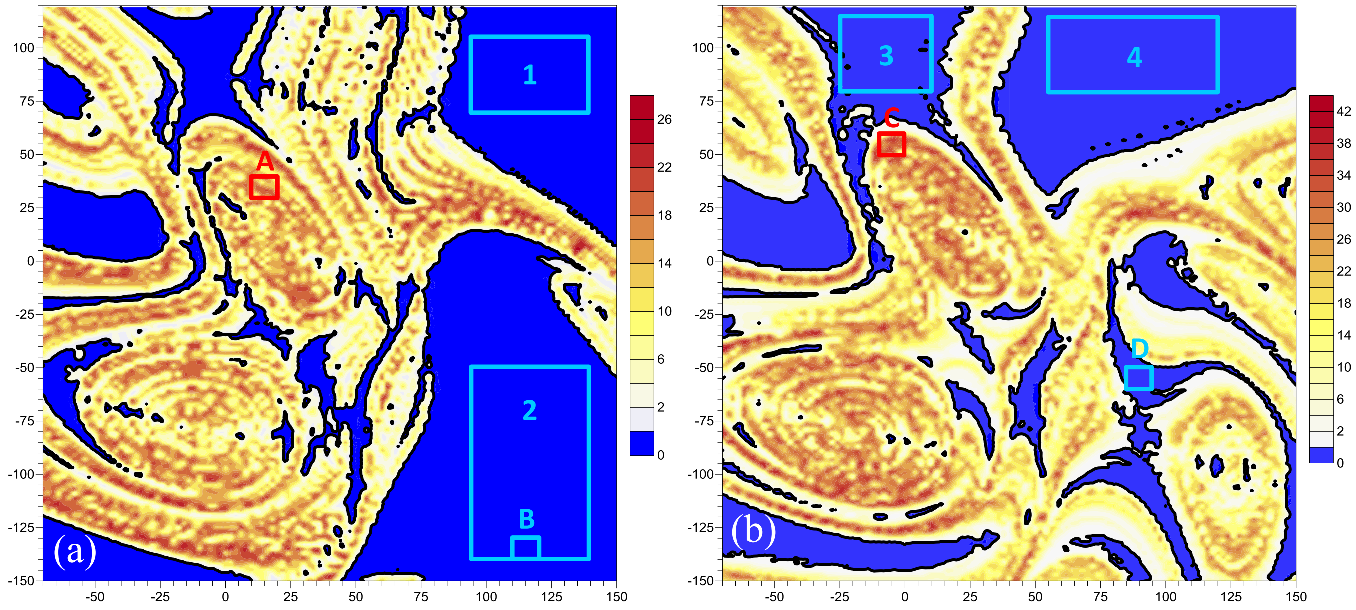


Pierini, Ghil & Chekroun
(*J. Clim.*, 2016)

Time-dependent forcing, IV

- There are strong numerical indications, along with theoretical justifications, that multiple PBAs are present within a global attractor.
- Moreover, preliminary numerical results suggest that the basin boundaries between two attractors are fractal.

Measure of divergence of trajectories for each initial point in the (ψ_1, ψ_3)-plane in the remote past: blue indicates stability; parameter values (left) and (right) are the same as in the previous figure.



Pierini, Ghil & Chekroun (*J. Clim.*, 2016)

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Concluding remarks

What do we know?

- There's an NAO, & it's important.
- It has decadal variability (7–8 yr).
- An oscillatory mode, albeit weak, can help prediction.
- Time-dependent forcing helps understand the coupled system.

What do we know less well?

- How does the climate system really work?
- Is it the tail that wags the dog —
  i.e., weather noise that drives a passive ocean?
- Or does the dog bite its tail –
  i.e., coupled O–A modes of decadal variability?
- Or does the old dog ocean plain wag its tail, the atmosphere?

What to do?

- Work the model hierarchy, and the observations!
- Explore further non-autonomous and randomly driven models,
  on the way to fully coupled ones.

Nature is not deterministic or stochastic:

It depends on what we can, need & want to know
— more or less detail, with greater or lesser accuracy —
larger scales more accurately,
smaller scales less so

But we need both, deterministic and stochastic descriptions.
Knowing how to combine them is necessary, as well as FUN!

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Reserve slides

Outline, Tipping Points I

Elementary Bifurcation Theory and Variational Principle

1. Fixed Points
   - linear stability
   - non-linear stability and attractor basins
2. Saddle-node bifurcations
   - multiple branches of stationary solutions
   - linear stability
3. Bifurcations in 1-D
4. Non-linear stability and variational principle
   - variational principle in 0-D
   - variational principle in 1-D
5. Bistability and hysteresis

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Concluding remarks, II

- Tipping points and bifurcations: do they really help?
  - Yes, if properly understood and carefully applied!
- Can we predict them?
  - Yes, depending on the problem and the data!

Spin-up of atmospheric jet

SST front:

L_{oc} = 600 km,

ΔT = 3.5 ^{0}C,

d = 50 km

Atmospheric jet

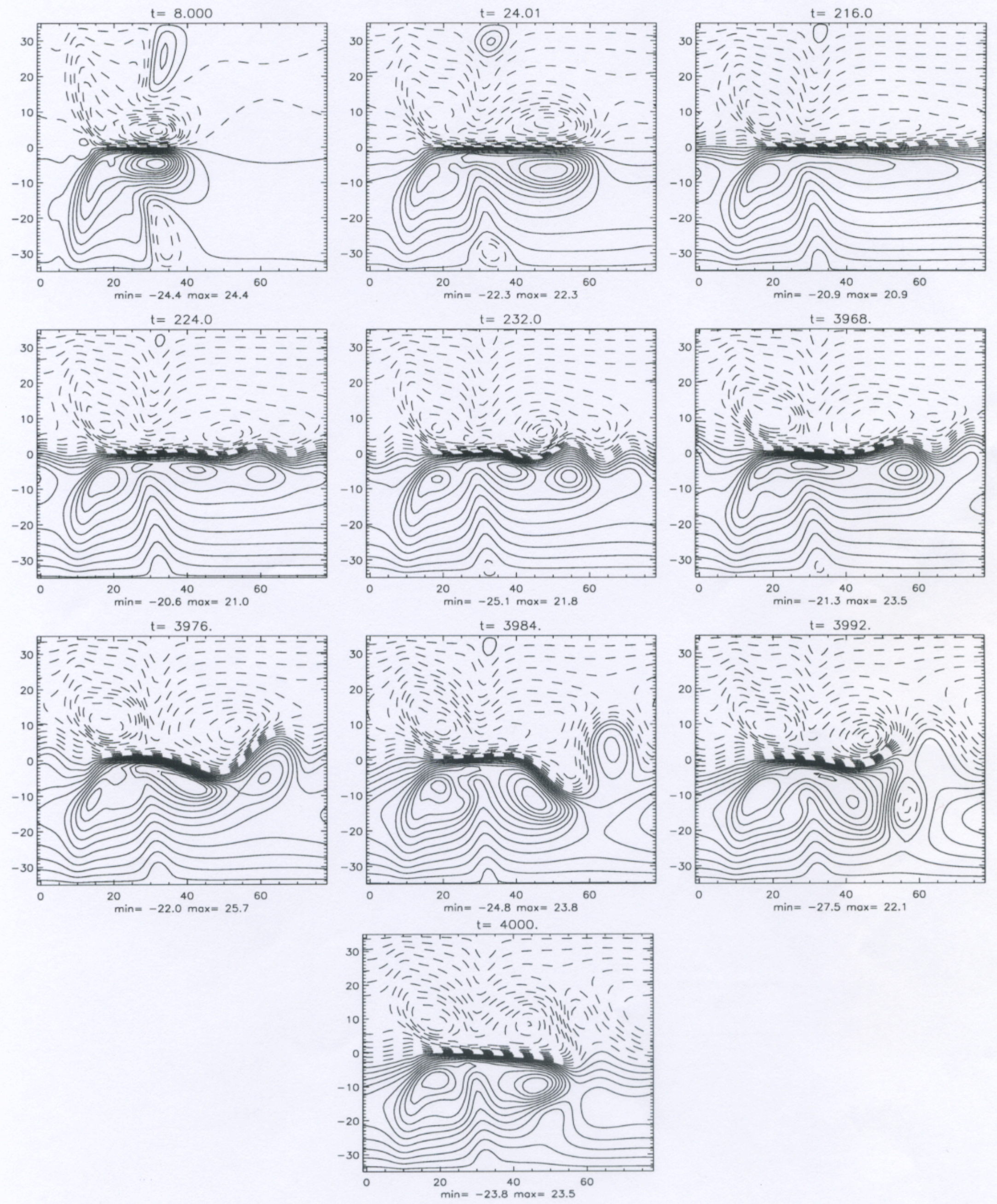
spins up from

L_a = 2000 km to

L_a = 4000 km, much

greater speed and

strong recirculation

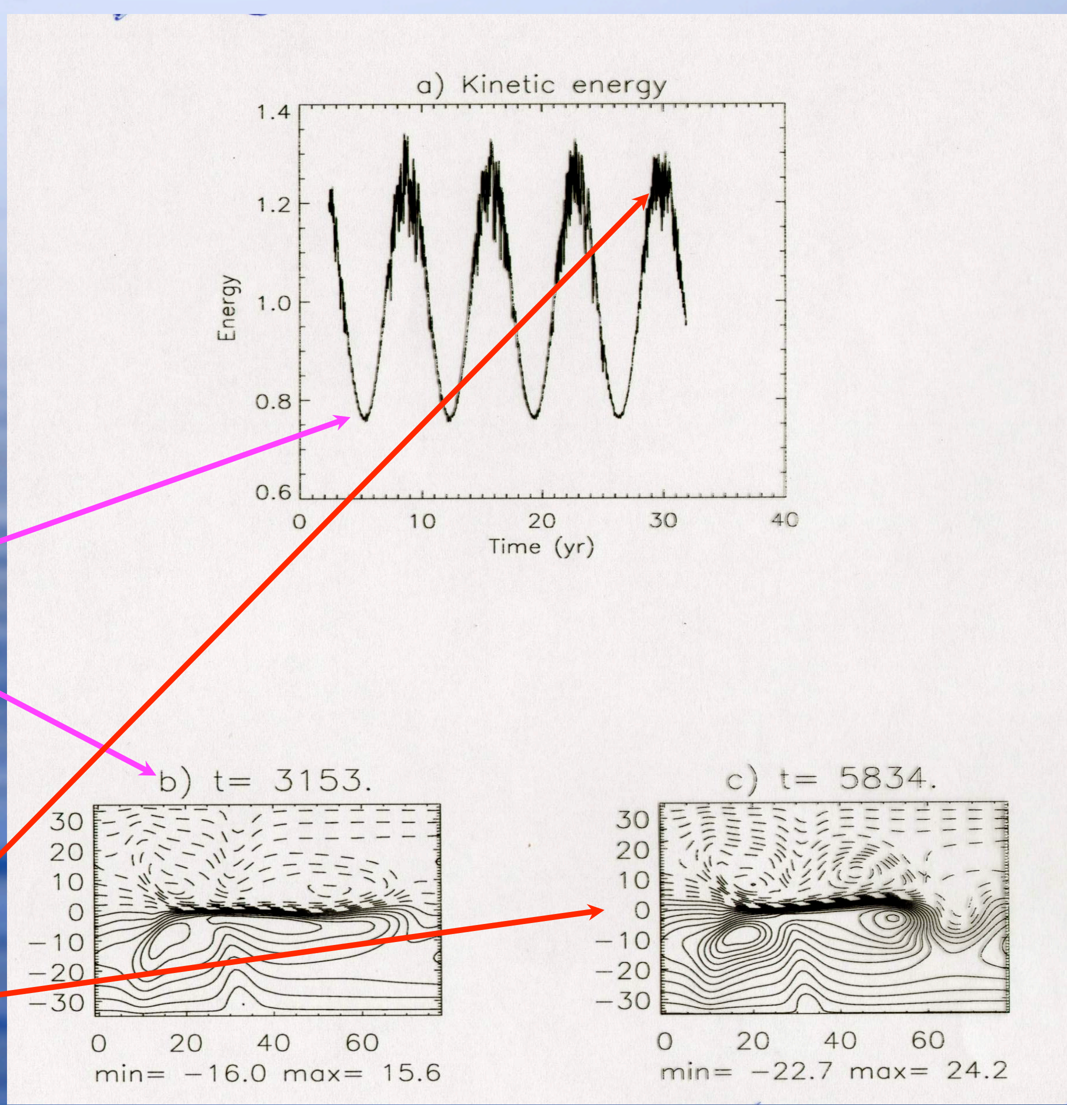


Forced 7-year cycle in the FGS'04 model

Slow amplitude modulation of 1 ^{0}C in the SST front

Low-energy phase

High-energy phase



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